Self-magnetized effects in relativistic cold plasmas due to ponderomotive forces: Application to relativistic magnetic guiding of light

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Nonlinear equations are derived relevant to describe the propagation of powerful electromagnetic fields launched within a plasma. The nonlinear generation of self-induced collective electromagnetic perturbations are obtained with matter lying in the relativistic regime. Our main result is the self-consistent treatment of the coupled equations between the pump and its self-induced fields. In particular, a mechanism is pointed out for self-generation of quasistatic magnetic field that is due to the relativistic ponderomotive force. This process is found to be more efficient to produce quasistatic magnetic fields, as confirmed by recent experiments, as compared to known effects such as the inverse Faraday effect. As an application, we investigate conditions for relativistic magnetic guiding of light to occur under the combined action of the self-induced density and magnetic field.

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INTRODUCTION

The recent achievement of powerful short laser pulses has stimulated a renewal of interest for radiation in the interaction with matter driven into relativistic regime. Proposals have emerged in the context of inertial fusion to use mixed (hybrid) confinement schemes [1], and also to help trigger the ignition in inertial confinement fusion by using self-induced transparency and hole boring [2]. The possible efficiency of these proposals relies mainly on the understanding of the mechanisms of propagation of the launched pump wave in the presence of self-generated magnetic fields. In the case of the hole boring scheme, the pump propagation starts into an initially overdense plasma.

Numerical simulations with particle in cell codes [3,4] in the overdense case have also confirmed the creation of strong self-generated magnetic fields and their important role. In the study of Ref. [3], the magnetic field was found to modify the filamentation process in the self-focusing relativistic case by causing the merging of filaments, thus favoring the guiding of light, while in a second study [4], the magnetic field seems to trap a few fast accelerated electrons that further deposit their energy on the bulk electron allowing to heat them efficiently.

Also in recent experiments with solid targets [5a], huge B fields have been measured around 10–100 MG and have been also predicted [5b].

These results and our previous studies [6–7] have motivated a deeper investigation of the mechanisms for slow magnetic-field creation in relativistic plasmas and the study of how these fields may influence the generating pump propagation. This paper is organized as follows. In the first part, we write the basic equations together with our working hypotheses. Then we solve, by a suitable perturbation expansion, the relativistic equations of motion for the momentum (and velocity) and for the associated vorticity. In the second part, we compute the generating nonlinear source currents for the induced-electromagnetic perturbations. Part three is devoted to the derivation of the *vectorial* coupled nonlinear evolution equations between pump and perturbations. In part four, we select an application from our general set to show different regimes for quasistatic self-magnetic fields evolution and their importance on the guiding of their pump wave.

Our main results in this paper may be summarized as follows. A perturbative treatment is used for finding solutions of the relativistic equation of motion for momentum and vorticity. Section III is derived of coupled evolution equations for the pump field in the presence of its self-generated magnetic field and of its induced-density perturbations. The proper evaluation of the couplings require the computation of the nonlinear source currents.

In the general case, we obtain four nonlinear coupled equations for the pump pulse, the self-induced density and magnetic field and for the vorticity Ω (when it does not vanish).

This general set of equations depends on various parameters that are identified. This system generates a high number of simplified subsets, obtained by taking suitable approximations.

Other nonlinear currents such as harmonics of the pump currents could be also important source terms in the overdense case. However, in this paper we shall restrict mainly to the underdense plasma case.

Next, we point out various relevant mechanisms for magnetic-field creation using a self-consistent approach in the collisionless plasma case. The ponderomotive source current term is emphasized, since it turns out to be especially important. As we shall see in previous studies, the contribution of the slow-frequency velocity has been neglected.

In Sec. IV, we investigate one subsystem of the general set that we deem to be important. We perform the quasistatic approximation (QSA hereafter) for the low-frequency perturbations density (n_S or $\langle n\rangle$) and magnetic field (B_S or $\langle B\rangle$) and show that the strongest source of self-induced B_S field comes here from ponderomotive effects.

The subsequent modification of the pump leads to relativistic magnetic guiding, i.e., a combination of relativistic self focusing (through n_S) and of magnetization (through B_S). The ponderomotive mechanism for B_S generation well explains, also in the nonrelativistic limit, recent experiments [8(a)–8(c)] whose interpretation has formerly relied incorrectly upon the inverse Faraday effect [8d].

I. PERTURBATIVE SOLUTIONS OF THE EQUATIONS OF MOTION

We start with the Maxwell equations for light (using a classical description) and we restrict it to the relativistic hydrodynamical equations for the plasma in the cold temperature case (i.e., pressure effects are neglected). Ions are assumed motionless forming a neutralization background. The plasma is assumed to be a collisionless gas underdense (unless specified) that is $\omega_{\mathrm{pe}}<\omega$, where ω_{pe} is the electron plasma frequency and ω the main pulse frequency. We note as $\delta=\omega_{\mathrm{pe}}/\omega$ their ratio.

We assume the existence of two time scales compatible with the underdense assumption: one scale is associated with the launched electromagnetic wave period $T=2\pi/\omega$ and the other one with the period of the self-induced perturbations $T=T_p,T_c$ (where $T_p=2\pi/\omega_{\mathrm{pe}}$, $T_c=2\pi/\omega_{\mathrm{ce}}$ are the electron plasma and cyclotron periods). These periods should be compared with the duration of the pulse τ (third time scale). In the envelope approximation we can distinguish the short and long pulse cases according, respectively, to the values of the ratios $[(\tau/T_c,\tau/T_p)<1$ or $(\tau/T_c,\tau/T_p)>1]$. We suppose it is also possible for the splitting of any quantity into fast- (hereafter index h) and slow-time variable (index $\langle\,\rangle$), the symbol $\langle\,\rangle$ meaning a time (or a phase) average over the fast-time scale.

A. Basic equations of motion

We concentrate first on the equations of motion and write them by introducing the generalized momentum $\tilde{P}$ and its related vorticity Ω. The electron charge is noted $q_e(=-e)$.

$$\vec{\Omega}=\mathrm{curl}(\vec{\tilde{\mathrm{P}}}),\tag{1}$$

$$\vec{\tilde{\mathrm{P}}}=(\vec{\mathrm{P}}+q_e\vec{\mathrm{A}}),\tag{2}$$

$$\vec{\mathrm{E}}=-\partial_t\vec{\mathrm{A}}-\vec{\nabla}\Phi,\tag{3}$$

$$\vec{\mathrm{B}}=\mathrm{curl}(\vec{\mathrm{A}}),\tag{4}$$

$$\partial_t\vec{\tilde{\mathrm{P}}}-\vec{V}\times\vec{\Omega}=-q_e\vec{\nabla}(m_0c^2\gamma+\Phi),\tag{5}$$

$$\partial_t\vec{\Omega}=\mathrm{curl}(\vec{\mathrm{V}}\times\vec{\Omega}),\tag{6}$$

$$\vec{\mathrm{V}}=\vec{\mathrm{P}}/m_0\gamma=(\vec{\tilde{\mathrm{P}}}-q_e\vec{\mathrm{A}})/m_0\gamma,\tag{7}$$

$$\gamma\equiv(1-V^2/c^2)^{-1/2}\quad\text{or}\quad\gamma^2=1+\vec{\mathrm{P}}\cdot\vec{\mathrm{P}}/(m_0c)^2.\tag{8}$$

Splitting the momentum $\vec{\mathrm{P}}$ into its slow and fast components we get on the fast time scale

$$\vec{\mathrm{P}}=\vec{\mathrm{P}}_h+\langle\vec{\mathrm{P}}\rangle,$$

$$\partial_t\vec{\tilde{\mathrm{P}}}_h=\vec{\mathrm{V}}_h\times\vec{\Omega}_h+\langle\vec{\mathrm{V}}\rangle\times\vec{\Omega}_h+\vec{\mathrm{V}}_h\times\langle\vec{\Omega}\rangle-q_e\vec{\nabla}(m_0c^2\gamma+\Phi)_h,\tag{9}$$

$$\partial_t\vec{\Omega}_h=\mathrm{curl}\ (\vec{\mathrm{V}}_h\times\vec{\Omega}_h+\langle\vec{\mathrm{V}}\rangle\times\vec{\Omega}_h+\vec{\mathrm{V}}_h\times\langle\vec{\Omega}\rangle),\tag{10}$$

and on the slow time scale

$$\partial_t\langle\vec{\tilde{\mathrm{P}}}\rangle=\langle\vec{\mathrm{V}}\rangle\times\langle\vec{\Omega}\rangle+\langle\vec{\mathrm{V}}_h\times\vec{\Omega}_h\rangle-\vec{\nabla}(m_0c^2\langle\gamma\rangle+q_e\langle\Phi\rangle),\tag{11}$$

$$\partial_t\langle\vec{\Omega}\rangle=\mathrm{curl}\langle\vec{\mathrm{V}}\rangle\times\langle\vec{\Omega}\rangle+\langle\vec{\mathrm{V}}_h\times\vec{\Omega}_h\rangle).\tag{12}$$

Thus, we have two nonlinear coupled equations for $\vec{\mathrm{P}}_h$ and $\langle\vec{\mathrm{P}}\rangle$ with Eqs. (9), (11), and two other equations for $\vec{\Omega}_h$ and $\langle\vec{\Omega}\rangle$ with (10), (12), since the velocity $\vec{V}$ and the Lorentz factor γ are themselves nonlinear functions of $\vec{\mathrm{P}}$.

It is difficult to deal with the full nonlinear case since only a few specific cases are completely solvable analytically. We shall perform a perturbative treatment into various parameters, in particular into the energy parameter defined as $q=E/E_C$, $E_C=(m_0c\omega/e)$ being the Compton electric field. To proceed, we "linearize" the motion (see the precise meaning below) on the fast-time scale and we keep as far as possible nonlinearities on the slow scale. We wish to study the modification of the fast scale by the slow self-induced perturbations.

Perturbative expansion. Any physical quantity S will be expanded into a series of powers of q as follows:

$$S=\sum_{l=0}^{l=+\infty}S^{(l)}q^l.$$

B. High-frequency motion

We work with Eqs. (9)–(10) by injecting the q expansion, but we restrict mainly to the linearized high frequency (hf) motions here

$$\vec{\mathrm{P}}_h/m_0=(\gamma\vec{\mathrm{V}}_h)=\langle\gamma\rangle\vec{\mathrm{V}}_h+\gamma_h\vec{\mathrm{V}}_h+\gamma_h\langle\vec{\mathrm{V}}\rangle,$$

$$\vec{\mathrm{P}}_h^{(3)}\approx m_0(\langle\gamma\rangle\vec{\mathrm{V}}_h)^{(3)}+\vartheta(q^n)n\geqslant 4.$$

The linearization of Eq. (9) gives (for $\Phi_h=0$) to order q (or for $\langle\vec{\Omega}\rangle=\vec{0}$): $\vec{\tilde{\mathrm{P}}}_h^{(1)}=\vec{0}$, to third order in q

$$\partial_t\vec{\tilde{\mathrm{P}}}_h\approx\vec{\mathrm{V}}_h\times\langle\vec{\Omega}\rangle,\tag{13}$$

a solution of Eq. (13) is (for $\vec{\mathrm{P}}_h$ and $\vec{\tilde{\mathrm{P}}}_h$ noncolinear)

$$\vec{\tilde{\mathrm{P}}}_h=\overleftrightarrow{\mathrm{U}}_{\langle\vec{\Omega}\rangle/m_0\langle\gamma\rangle,\omega}(\vec{\mathrm{A}}_h\times\langle\vec{\Omega}\rangle)(-q_e/m_0\langle\gamma\rangle),\tag{14a}$$

$$\vec{\mathrm{V}}_h^{(3)}=\overleftrightarrow{\mathrm{U}}_{\vec{\Omega}/m_0\langle\gamma\rangle,\omega}(\vec{\mathrm{A}}_h^{(m)})(-q_e/m_0\langle\gamma\rangle^{(n)})|_{m+n=3}.\tag{14b}$$

The operator $\overleftrightarrow{U}$ is defined in order to invert the cyclotron motion of frequency ω_C by

$$-i\omega\vec{A}+\vec{\omega}_c\times\vec{A}=\vec{S}:\vec{A}=\overleftrightarrow{U}_{\omega,\omega_c}(\vec{S})/(-i\omega),$$

$$\overleftrightarrow{U}_{\omega,\omega_c}(\vec{S})=\{\overleftrightarrow{I}_{d_\parallel}+1/\Delta[\overleftrightarrow{I}_{d_\perp}-(\vec{\omega}_c/-i\omega)]\times\overleftrightarrow{I}_{d_\perp}\}\vec{S}, \quad (15)$$

$$\Delta=1/[1-(\omega_c/\omega)^2], \quad \vec{\omega}_c=q_e\vec{B}/m_0.$$

We make the same approximations for solving Eq. (10) as for Eq. (9), that is to order q, we have $\vec{\Omega}_h^{(1)}=\vec{0}$, since $\vec{V}_h^{(1)}$ from (14b) reduces to Eq. (17) below. To third order in q, we have instead

$$\partial_t\vec{\Omega}_h{}^{(3)}\approx \mathrm{curl}(\vec{V}_h^{(1)}\times\langle\vec{\Omega}^2\rangle, \quad (16')$$

$$\rightarrow\vec{\Omega}_h^{(3)}\approx \mathrm{curl}(\vec{r}_h^{(1)}\times\langle\vec{\Omega}^{(2)}\rangle) \quad (16)$$

with $\partial_t\vec{r}_h=\vec{V}_h$,

$$\vec{V}_h^{(1)}(q)=-q_e\vec{A}_h^{(1)}/[m_0(\langle\gamma\rangle^{(0)})], \quad (17)$$

$$|\vec{V}_h^{(1)}|/c\equiv q.$$

Here, we have taken the velocity V_h to the first order only and we have assumed that $\langle\vec{\Omega}\rangle$ is of at least second order in q.

To go from Eq. (16′) to Eq. (16), we also suppose that the parameter δ is small enough, implying that $\langle\vec{\Omega}\rangle$ remains constant on the fast-time scale, besides the evolution of $\langle\vec{\Omega}\rangle$ gives a term that scales similar to δq^3.

It is easy also to show that the fast part of the Lorentz factor γ_h term starts in q^2 at twice the pump frequency 2ω and that its next term scales like q^3 at frequencies ω and 3ω. Then, the $-m_0c^2\vec{\nabla}\gamma_h$ expression should be kept in the right-hand side (rhs) of Eq. (9b) at the same level than the $\vec{V}_h^{(1)}\times\langle\vec{\Omega}\rangle$ contribution, but it scales like αq^3, with α being a spatial parameter defined by the ratio of the wavelength λ to the electric-field gradient L_e: $\alpha=2\alpha\pi\lambda/L_e$. These two terms may be comparable. However, when the gradient is applied on γ_h it vanishes for a transverse pump wave, thus one can neglect this contribution to the motion at the third order. The next relevant term in $(\langle\vec{V}\rangle\times\vec{\Omega}_h)$ is of fifth order in q.

The solutions from Eqs. (14) are thus correct approximations to third order in q. In general we have an expansion of the variables into a triple series of the three parameters q, α, and δ [7].

The high-frequency (hf) equation of motion is thus ‘‘linearized’’ in the sense that it contains only the slow scale as a nonlinear factor and no more the hf terms themselves of the same order.

The computation is self consistent because, for example, $\langle\vec{P}\rangle$ to second order in q will depend on terms involving $\langle\vec{V}_h^{(1)}\cdot\vec{V}_h^{*(1)}\rangle$ already known to the lowest order in q. (Rejecting the slow terms in fast equations allows us to make a closure between high- and low-frequency motions.)

The computation of second-order terms such as $\vec{P}_h^{(2)}$ and of its time average are described shortly in the next section, since such a term will contribute to the slow momentum $\langle\vec{P}\rangle^{(2)}$ as ponderomotive sources.

C. Low-frequency motion

We use Eqs. (11) and (12), the main hf quadratic source terms (of lowest order in q) being the one in $\langle\vec{V}_h\times\vec{\Omega}_h\rangle$; with the linearized velocity $\vec{V}_h^{(1)}$ and $\vec{\Omega}_h$ given by Eq. (16) we have (to fourth order in q)

$$\langle\vec{V}_h\times\vec{\Omega}_h\rangle=-\vec{V}_r\times\langle\vec{\Omega}\rangle-f(\langle\vec{\mu}\rangle), \quad (18)$$

$$\vec{V}_r\equiv\langle(\vec{V}_h\cdot\vec{\nabla})\vec{r}_h\rangle[=-\langle(\vec{r}_h\cdot\vec{\nabla})\vec{V}_h\rangle], \quad (19)$$

$$\langle\vec{\mu}\rangle\equiv(q_e/2)\langle\vec{r}_h\times\vec{V}_h\rangle, \quad (20)$$

$$f\equiv\vec{\nabla}[\langle\vec{\Omega}\rangle\cdot(\langle\vec{\mu}\rangle/q_e)], \quad (21)$$

one recognizes in $(-m_0c^2\vec{\nabla}\langle\gamma\rangle-f)$ the expansion of $(-m_0c^2)\vec{\nabla}[(\langle\gamma'\rangle(\vec{\mu})]$

$$\langle\gamma'\rangle=\{\gamma_0^2+(2\langle\vec{\Omega}\rangle\cdot(\langle\vec{\mu}\rangle/q_e)/(m_0c^2\}^{1/2}, \quad (22)$$

$$[\gamma_0\equiv\langle\gamma(\vec{\mu}=\vec{0})\rangle].$$

We have introduced the magnetic moment $\langle\vec{\mu}\rangle$ with Eq. (20) and a renormalized velocity $\vec{V}_r$ by Eq. (19) that depends on the high-frequency displacement r_h, in a guiding center approach of the fluid. From Eq. (22), we recover the known nonrelativistic result at low q for the ponderomotive potential energy, with the pseudo (inverse sign as compared to true energy) ‘‘generalized’’ magnetic energy (because we use the expression $\langle\vec{\Omega}\rangle\cdot\langle\vec{\mu}\rangle/q_e$ instead of $\langle\vec{B}\rangle\cdot\langle\vec{\mu}\rangle$) given by

$$m_0c^2(\langle\gamma'\rangle-1)\approx[m_0(\vec{V}_h^{(1)})^2/2]+\langle\vec{\mu}\rangle\cdot\langle\vec{\Omega}\rangle/q_e. \quad (23)$$

For a single particle, there is no induced slow field $\langle\vec{B}\rangle$ and one recovers the following expression of the energy, if an external field B_0 is applied, as

$$m_0c^2(\langle\gamma'\rangle-1)\approx[m_0(V_h{}^{(1)})^2/2]+\langle\vec{\mu}\rangle\cdot\vec{B}_0. \quad (24)$$

Note that expressions (22)–(24) already hold terms to fourth order in q. Another useful relation is the following:

$$\langle\mathrm{curl}(\vec{r}_h\times\vec{V}_h)\rangle=2\langle-\vec{V}_h\vec{\nabla}\cdot(\vec{r}_h)\rangle+2\vec{V}_r. \quad (25)$$

Equation (25) is valid for time-averaged periodic variables with zero mean values.

So far, the ponderomotive term defined by Eq. (22) derives from the gradient of a potential only. However, there is also a rotational (curl) contribution to the average ponderomotive energy, whose contribution could be important for magnetic-field generation in the short- but also in the long-pulse case.

Relying on the results we have derived elsewhere [9], it can be shown that the full ponderomotive fourth-order potential $(\vec{A}_p, \Phi_p)$ should be considered with, hence, a curl part of the ponderomotive force. For example, in the fluid case (labeled with an index f) we have a ponderomotive force $\vec{\mathrm{F}}_p$ as

$$\vec{\mathrm{F}}_p/q_e = -\partial_t \vec{\mathrm{A}}_{p,f} - \vec{\nabla}\Phi_{p,f}. \tag{26}$$

The associated self-consistent force should include an additional term as (to fourth order in q)

$$\vec{\mathrm{F}}_p/q_e = -\partial \vec{\mathrm{A}}_{p,f} - \vec{\nabla}\Phi_{p,f} + \langle \vec{\mathrm{V}} \rangle \times \mathrm{curl}(\vec{\mathrm{A}}_{p,f}). \tag{26$'$}$$

By evaluating the average of the fast-fluid momentum $\langle \vec{\mathrm{P}}_h^{(2)} \rangle$ to second order, one can find a connection [9] between the fluid vector ponderomotive potential $\vec{\mathrm{A}}_{p,f}$ and the velocity $\vec{\mathrm{V}}_r$ defined by Eq. (19) as follows:

$$q_e \Phi_{p,f} = -m_0 c^2 (\langle \gamma' \rangle - 1), \tag{27}$$

$$q_e \vec{\mathrm{A}}_{p,f} = m_0 \langle \gamma \rangle \vec{\mathrm{V}}_r, \tag{28}$$

$$-\partial_t \vec{\mathrm{A}}_{p,f}^{(2)} - \vec{\nabla}\Phi_{p,f} \approx -m_0 c^2 \vec{\nabla}(\langle \gamma \rangle - 1) + \partial_t \langle (\vec{r}_h{}^{(1)} \cdot \vec{\nabla}) \vec{\mathrm{P}}_h^{(1)} \rangle. \tag{29}$$

From Eqs. (26′) and (29) it can be seen that a curl contribution enters into the ponderomotive force in addition to the longitudinal part (the gradient of the ponderomotive potential). This second term in the rhs of Eq. (29) is generally in the ratio $\omega\tau$ with respect to the first one. The nonrelativistic limit is easily recovered by expanding the Lorentz factor in powers of $(v/c)^2$.

Now we wish to solve the slow equations of motion (11) and (12) at several levels. This computation is important since the involved velocities are building the source terms responsible for the self-magnetic-field generation processes and for the self-induced density as well.

1. General case

Equation (11) is a nonlinear equation yielding for example the velocity $\langle \vec{\mathrm{V}} \rangle$. Its Fourier transform at a frequency ω_0 is given by Eq. (31)

$$\langle \vec{\mathrm{P}} \rangle = m_0 (\langle \vec{\mathrm{V}} \rangle \langle \gamma \rangle + \langle \vec{\mathrm{V}}_h \gamma_h \rangle), \tag{30}$$

$$\langle \vec{\mathrm{V}} \rangle(\omega) \approx \vec{\vec{\mathrm{U}}}_{\langle \Omega \rangle / m_0, \omega_0} \{ \vec{\mathrm{S}}_1 / (-i\omega_0) - q_e \langle \vec{\mathrm{A}} \rangle \} / (m_0 \langle \gamma \rangle), \tag{31}$$

$$\vec{\mathrm{S}}_1 = \langle \vec{\mathrm{V}}_h \times \vec{\Omega}_h \rangle - \vec{\nabla}(m_0 c^2 \langle \gamma \rangle + q_e \langle \Phi \rangle). \tag{32}$$

Where $\vec{\Omega}_h$ is given, for example, by Eq. (16) to third order in q for $\delta < 1$. Here, ω_0 is an oscillation (for example ω_C/γ) on the slow-time scale. More generally, we may consider Eq. (11) as a fluid self-consistent equation, with the ponderomotive contribution included, by writing Eq. (33) as

$$\partial_t(\langle \vec{\mathrm{P}} \rangle) = -q_e \partial_t (\langle \vec{\mathrm{A}}' \rangle - \vec{\mathrm{V}}_r \times \langle \vec{\Omega} \rangle - \vec{\nabla}(\psi') + \langle \vec{\mathrm{V}} \rangle \times \langle \vec{\Omega}' \rangle, \tag{33}$$

$$\langle \vec{\mathrm{A}}' \rangle \equiv \langle \vec{\mathrm{A}} \rangle + \vec{\mathrm{A}}_p, \tag{34a}$$

$$\langle \vec{\Omega}' \rangle \equiv \langle \vec{\Omega} \rangle + q_e \,\mathrm{curl}(\vec{\mathrm{A}}_p), \tag{34b}$$

$$\Psi' \equiv m_0 c^2 (\langle \gamma' \rangle - 1) + q_e \langle \Phi \rangle. \tag{35}$$

γ' is given by Eq. (22), and using Eq. (30) we get

$$\langle \vec{\mathrm{V}} \rangle + \langle \vec{\mathrm{V}} \rangle \times \langle \vec{\Omega}' \rangle / (i\omega_0 m_0 \langle \gamma \rangle) \approx \vec{\mathrm{S}}_2 / (-i\omega_0 m_0 \langle \gamma \rangle), \tag{36}$$

$$\vec{\mathrm{S}}_2 = \vec{\mathrm{S}}_{2'1} - i\omega_0 m_0 \langle \gamma \rangle (\vec{\mathrm{V}}'_0 - \vec{\mathrm{V}}_1), \tag{37}$$

$$\vec{\mathrm{S}}_{2'1} = -\vec{\mathrm{V}}_r \times \langle \vec{\Omega} \rangle - \vec{\nabla}(\psi'),$$

$$\begin{aligned}\langle \vec{\mathrm{V}} \rangle = {} & \vec{\vec{\mathrm{U}}}_{\omega_0, \langle \Omega' \rangle / m_0 \langle \gamma \rangle} (\vec{\mathrm{S}}_{2,1}) / (-i\omega_0 m_0 \langle \gamma \rangle) \\ & + \vec{\vec{\mathrm{U}}}_{\omega_0, \langle \Omega' \rangle / m_0 \langle \gamma \rangle} (\vec{\mathrm{V}}'_0 - \vec{\mathrm{V}}_1),\end{aligned} \tag{38}$$

$$\vec{\mathrm{V}}'_0 \equiv -q_e \langle \vec{\mathrm{A}}' \rangle / (m_0 \langle \gamma \rangle), \tag{39a}$$

$$\vec{\mathrm{V}}_1 \equiv -(\langle \vec{\mathrm{V}} \rangle \partial_t \langle \gamma \rangle + \partial_t \langle \vec{\mathrm{V}}_h \gamma_h \rangle / \langle \gamma \rangle. \tag{39b}$$

2. *l* and *t* components of a vector

The longitudinal (l) and transversal (t) components of a vector are defined by

$$\vec{\mathrm{V}} = \vec{\mathrm{V}}^l + \vec{\mathrm{V}}^t,$$

$$\vec{\mathrm{V}}^l : \vec{\nabla} \cdot (\vec{\mathrm{V}}^l) \neq 0, \ \ \mathrm{curl}(\vec{\mathrm{V}}^l) = 0,$$

$$\vec{\mathrm{V}}^t : \vec{\nabla} \cdot (\vec{\mathrm{V}}^t) = 0, \ \ \mathrm{curl}(\vec{\mathrm{V}}^t) \neq 0.$$

To extract these components in the inversion of the cyclotron motion, we make the useful approximations that become exact for cylindrical geometry and for $\Omega' = \mathrm{const}$; for a given vector S, we have the rules

$$(\vec{\mathrm{S}} \times \vec{\Omega})^l \cong \vec{\mathrm{S}}^t \times \vec{\Omega}, \tag{40a}$$

$$(\vec{\mathrm{S}} \times \vec{\Omega})^t \cong \vec{\mathrm{S}}^l \times \vec{\Omega}. \tag{40b}$$

Application to the velocity components is given as

$$\langle \vec{\mathrm{V}} \rangle^l = \vec{\vec{\mathrm{U}}}_{\langle \Omega' \rangle / m_0 \langle \gamma \rangle, \omega_0} (\vec{\mathrm{S}})^l / (-i\omega_0 m_0 \langle \gamma \rangle), \tag{41}$$

$$\langle \vec{\mathrm{V}} \rangle^l \approx \{ \vec{\vec{H}}(\vec{\mathrm{S}}^l) - i\vec{\mathrm{S}}^t \times \langle \vec{\Omega}' \rangle / (\omega_0 m_0 \Delta \langle \gamma \rangle) \},$$

$$\vec{\vec{H}}(\cdot) \equiv ((\vec{\vec{I}}_{d\perp}/\Delta) + I_{d\parallel})(\cdot), \tag{42}$$

$$\langle \vec{\mathrm{V}} \rangle^t = \vec{\vec{\mathrm{U}}}_{\langle \Omega' \rangle / m_0 \langle \gamma \rangle, \omega_0} (\vec{\mathrm{S}}')^t / (-i\omega_0 m_0 \langle \gamma \rangle,$$

$$\langle \vec{V} \rangle^t \approx \{ (H(\vec{S}'^t) - i\vec{S}'^l \times \langle \vec{\Omega}' \rangle / (\omega_0 m_0 \Delta \langle \gamma \rangle) \}. \quad (43)$$

It could be seen from Eqs. (41)–(43) that the velocity may have longitudinal components. These components have been neglected, for example, in a former study about self-magnetic-field generation induced by circularly polarized pulses [8(d)]. We shall see that these longitudinal velocities generate a stronger amplitude of the slow-magnetic field than the transverse components of the velocity, in particular, in the case of the inverse Faraday effect (IFE).

D. The quasistatic approximation

One could make the quasistatic approximation (QSA hereafter) that consists in neglecting the slow-time scale variation ($dt_S = 0$) in order to describe quasistationary processes. The QSA is valid for long enough pulses $\omega\tau \gg 1$ (see the envelope approximation section). In this limit, we can derive the following solutions for the slow motion. From Eq. (33) we have

$$\partial_{ts} \langle \vec{\tilde{P}}' \rangle \approx \vec{0} \Rightarrow \langle \vec{V} \rangle = \vec{V}_r + \langle \vec{\Omega}' \rangle \times \vec{\nabla}(\Psi')/|\vec{\Omega}'|^2. \quad (44)$$

Special cases from Eq. (33) are as follows:

(a) Case $\vec{\Omega}_p = \vec{0}$ (subscases $\langle \vec{\mu} \rangle = \vec{0}$ or $\langle \vec{\mu} \rangle \neq \vec{0}$). We have

$$(\langle \vec{V} \rangle - \vec{V}_r) \times \langle \vec{\Omega} \rangle - \vec{\nabla} \Psi' \approx \vec{0}$$

with the simple solution

$$\langle \vec{V} \rangle = \vec{V}_r + (\vec{\nabla} \varphi), \quad (45)$$

$$\vec{\nabla} \Psi' = \vec{0} \quad \text{or} \quad q_e \langle \Phi \rangle|_{q^4} = -m_0 c^2 (\langle \gamma' \rangle - 1). \quad (46')$$

A simple subcase often considered but not self consistent to third order in q consists in neglecting $\vec{V}_r$ in Eq. (45) and we are left with

$$\langle \vec{V} \rangle = \vec{0} \quad \textit{and} \quad \langle \Phi \rangle|_{q^4} = -m_0 c^2 (\langle \gamma' \rangle - 1). \quad (46)$$

Equation (46′) appears as a necessary compatibility condition.

(b) Case $\vec{\Omega}_p \neq \vec{0}$ (subcases $\langle \vec{\mu} \rangle = \vec{0}$ and $\langle \vec{\mu} \rangle \neq \vec{0}$). In this last case, Eq. (44) yields the general solution for $\langle \vec{V} \rangle$. Now as a check, we may take the QSA limit from the general case by letting ω_0 going to zero and from Eqs. (41)–(43) one recovers Eq. (44) with $\vec{S}$ given by Eq. (37).

$$\lim_{\omega_0 \to 0} (\langle \vec{V} \rangle) = \langle \vec{\Omega}' \rangle / |\langle \Omega' \rangle|^2 \times \vec{S} + \lim_{\omega_0 \to 0} (\vec{S}_\parallel - \vec{S}_\perp / \Delta)/(-i\omega_0 m_0 \langle \gamma \rangle). \quad (47)$$

However, one cannot yet conclude whether $\vec{\nabla} \Psi' \neq \vec{0}$ or not in this limit, and we need a more precise computation, to be done in the next section.

II. COMPUTATION OF THE NONLINEAR CURRENTS

To proceed, we use now the Maxwell equations that are coupled with the hydrodynamical equations and we implement the results of Sec. I on the velocities, to find the current sources responsible for $\langle n \rangle$ and $\langle B \rangle$ generation processes.

A. On the fast-time scale

The velocity $\vec{V}_h$ being computed on the fast scale, we can find the density perturbation n_h by using the continuity equation and the fast varying current $\vec{J}_h$ as

$$\partial_t n + \vec{\nabla} \cdot (n\vec{V}) = 0, \quad (48)$$

$$\partial_t n_h + \vec{\nabla} \cdot (\vec{J}_h / q_e) = 0,$$

$$\vec{J}_h / q_e = \langle n \rangle \vec{V}_h + n_h \langle \vec{V} \rangle + n_h \vec{V}_h. \quad (49)$$

1. Expansion for density perurbation

To first order in q, by linearizing the continuity equation, we find the density $n_h^{(1)}$ as

$$\partial_t n_h^{(1)} \approx -\vec{\nabla} \cdot (n_0 \vec{V}_h^{(1)}) + \vartheta(q^3),$$

$$n_h^{(1)} \approx -\vec{\nabla} \cdot (n_0 \vec{r}_h^{(1)}). \quad (50)$$

To third order in q, one should compute the current $\vec{J}_h$:

$$\partial_t n_h^{(3)} = -\vec{\nabla} \cdot (\vec{J}_h^{(3)})/q_e,$$

$$\vec{J}_h^{(3)}/q_e = n_0 \vec{V}_h^{(3)} + \langle n \rangle^{(2)} \vec{V}_h^{(1)} + n_h^1 \langle \vec{V} \rangle^{(2)} + (n_h \vec{V}_h)^{(3)}, \quad (51)$$

for $\delta < 1$

$$n_h^{(3)} \approx -\vec{\nabla} \cdot \bigg(\langle n \rangle^{(2)} \vec{r}_h^{(1)} + \langle \vec{V} \rangle^{(2)} \int n_h^{(1)} dt + n_0 \vec{r}_h^{(3)} + \int (n_h \vec{V}_h)^{(3)} (dt) + \vartheta(\delta q^3), \quad (51')$$

with $n_h^{(1)}$ given by Eq. (50). The Eqs. (51) and (51′) involve currents at the frequency harmonics that will be described in Sec. III A.

B. On the slow-time scale

The suitable equation for deriving the potential term ($\langle n \rangle$ or $\langle \Phi \rangle$) on the slow-time scale is the longitudinal (l) part of the Maxwell-Ampere equation, see Eq. (54), instead of the hydrodynamical slow-time continuity equation (that is identically satisfied)

$$\text{curl}(\vec{B}) = \mu_0 (\vec{J} + \vec{J}_d) \quad (52)$$

with

$$\vec{J}_d = \varepsilon_0 \partial_t \vec{E},$$

$$\langle\vec{J}\rangle/q_e=\langle n\rangle\langle\vec{V}\rangle+\langle n_h\vec{V}_h\rangle, \tag{53}$$

$$\mathrm{curl}(\langle\vec{B}\rangle)=\mu_0(\langle\vec{J}\rangle^t+\langle\vec{J}_d\rangle^t),$$

$$\langle\vec{J}\rangle^l+\langle\vec{J}_d\rangle^l=0. \tag{54}$$

The slow current $\langle\vec{J}\rangle$ is defined by Eq. (53); it is made of two terms, which are (i) a self-consistent one noted as $\langle\vec{J}\rangle_{sc}$ and (ii) a source contribution coming from the average of high-frequency components noted as $\langle\vec{J}_h\rangle=\vec{J}_{\mathrm{slow}}$:

$$\langle\vec{J}\rangle_{sc}/q_e=\langle n\rangle\langle\vec{V}\rangle, \tag{55a}$$

$$\langle\vec{J}_h\rangle=\vec{J}_{\mathrm{slow}}/q_e=\langle n_h\vec{V}_h\rangle. \tag{55b}$$

Remark. In Eq. (52), we have assumed *a priori* no magnetization $\vec{M}$ and have identified the magnetic-field B with the magnetic induction $\vec{H}$. But, in general, we should write the Maxwell-Ampere equation on $\vec{H}$ and make use of the relation $\vec{B}=\mu_0(\vec{M}+\vec{H})$ as for metals. However, for plasmas, the $\vec{M}$ term will come out directly from the slow varying current (see below and Refs. [9,10]).

Now, using Eq. (43) for $\langle\vec{V}\rangle^1$ together with Eq. (37) for the longitudinal source term $\vec{S}_2{}^1$ and the prescription (40), we can get the complete expressions for the potentials and the velocities at the slow-time scale

$$\begin{aligned}\vec{\nabla}_{\parallel}\Psi'=a(&\langle n_h\vec{V}_h\rangle_{\parallel}^l+\langle n\rangle[(\vec{V}_0-\vec{V}_1)_{\parallel}^l\\&+(\vec{V}_r^t\times\langle\vec{\Omega}\rangle_{\parallel}/(i\omega_0 m_0\langle\gamma\rangle)]\\&+\vec{\nabla}_{\parallel}(m_0c^2\langle\gamma'\rangle/d),\end{aligned} \tag{56}$$

$$a=(-e^2/d),\quad d=[1-\omega_{pe}^2/(\omega_0^2\langle\gamma\rangle)].$$

$$\begin{aligned}\vec{\nabla}_{\perp}\Psi'=b\{&\langle n_h\vec{V}_h\rangle_{\perp}^l+\langle n\rangle/(1-\Delta)[(\vec{V}_0-\vec{V}_1)_{\perp}^l\\&+(\vec{V}_r\times\langle\vec{\Omega}\rangle)_{\perp}/(i\omega_0 m_0\langle\gamma\rangle)]\\&+\ldots\langle n\rangle/(1-\Delta)[i(\vec{V}_0-\vec{V}_1)_{\perp}^t\\&+(\vec{V}_r^l\times\langle\vec{\Omega}\rangle)_{\perp}/(\omega_0 m_0\langle\gamma\rangle)]\\&\times\langle\vec{\Omega}'\rangle/(\omega_0 m_0\langle\gamma\rangle)\}+\cdots+\nabla_{\perp}(m_0c^2\langle\gamma'\rangle)\\&\times(1-\Delta)/(1-\omega_{\mathrm{uh}}^2/\omega_0^2),\end{aligned} \tag{57}$$

$$b=(-e^2/i\epsilon_0\omega_0)/[1-\omega_{\mathrm{pe}}^2/(\omega_0^2\langle\gamma\rangle(1-\Delta))].$$

$$\omega_{\mathrm{uh}}^2=\omega_{\mathrm{pe}}^2/(\langle\gamma\rangle)+\langle\Omega'\rangle^2/(m_0\langle\gamma\rangle)^2 \tag{58}$$

for $\langle\Omega'\rangle/m_0\approx\omega_{\mathrm{ce},}$ we get

$$\omega_{\mathrm{uh}}^2=\omega_{\mathrm{pe}}^2/\langle\gamma\rangle+(\omega_{\mathrm{ce}}/\langle\gamma\rangle)^2.) \tag{58$'$}$$

In Eq. (58′) appears the relativistic upper-hybrid frequency for the electron. To compute the final expressions for the slow components of the velocities $\langle\vec{V}\rangle^1$ and $\langle\vec{V}\rangle^t$ we use again Eqs. (41)–(43) and the above results (56) and (57) for the potential Ψ' (valid to the fourth order in q), to get

$$\begin{aligned}\langle\vec{V}\rangle^l\approx H[&(\vec{V}_0-\vec{V}_1)^l+(\vec{V}_r^t\times\langle\vec{\Omega}\rangle+\vec{\nabla}\Psi')/(i\omega_0 m_0\langle\gamma\rangle)]+\cdots\\&+1/(1-\Delta)[(\vec{V}_0-\vec{V}_1)^t\\&+(\vec{V}_r^l\times\langle\vec{\Omega}\rangle/(i\omega_0 m_0\langle\gamma\rangle)]\times\langle\vec{\Omega}'\rangle)/(\omega_0 m_0\langle\gamma\rangle),\end{aligned} \tag{59}$$

$$\begin{aligned}\langle\vec{V}\rangle^t\approx &H(\vec{1}+\vec{2}+\vec{3})+1/(1-\Delta)(\vec{V}_r^t\times\langle\vec{\Omega}\rangle)\times\langle\vec{\Omega}'\rangle)+\cdots\\&+b\{\langle n_h\vec{V}_h\rangle_{\perp}^l+\langle n\rangle/(1-\Delta)[(\vec{V}_0-\vec{V}_1)_{\perp}^l\\&+(\vec{V}_r^t\times\langle\vec{\Omega}\rangle)_{\perp}/(i\omega_0 m_0\langle\gamma\rangle+\cdots+\langle n\rangle/(1-\Delta)\\&\times(i(\vec{V}_0-\vec{V}_1)_{\perp}^t+(\vec{V}_r^l\times\langle\vec{\Omega}\rangle)_{\perp}/(\omega_0 m_0\langle\gamma\rangle)]\\&\times\langle\vec{\Omega}'\rangle/(\omega_0 m_0\langle\gamma\rangle)+\cdots\langle\vec{\Omega}'\rangle/(1-\Delta)(\omega_0 m_0\langle\gamma\rangle^2\\&+[\vec{\nabla}_{\perp}(m_0c^2\langle\gamma'\rangle)/(1-\omega_{\mathrm{uh}}^2/\omega_0^2)]\\&\times\langle\vec{\Omega}'\rangle/(\omega_0 m_0\langle\gamma'\rangle)^2.\end{aligned} \tag{60}$$

$$\vec{1}=(\vec{V}_0-\vec{V}_1)^t,$$

$$\vec{2}=(\vec{V}_r^l\times\langle\vec{\Omega}\rangle)/(-i\omega_0 m_0\langle\gamma\rangle),$$

$$\vec{3}=i(\vec{V}_0-\vec{V}_1)^l\times\langle\vec{\Omega}'\rangle/[\omega_0 m_0\langle\gamma\rangle(1-\Delta)].$$

2. QSA limit for the slow velocity

It is interesting to take the QSA limit for both slow potential and slow velocities. This limit yields for the potentials given by Eqs. (56) and (57) the final results

$$\lim_{\omega_0\to 0}\vec{\nabla}_{\parallel}\Psi'=\vec{0}, \tag{61a}$$

$$\lim_{\omega_0\to 0}\vec{\nabla}_{\perp}\Psi'=\vec{\nabla}_{\perp}[(m_0c^2\gamma')(\langle\Omega'\rangle)^2/\omega_{\mathrm{uh}}^2(m_0\gamma)^2]\neq 0. \tag{61b}$$

With these expressions (61), we may compute in the same limit the velocity $\langle\vec{V}\rangle$ to get

$$\begin{aligned}\lim_{\omega_0\to 0}\langle\vec{V}\rangle=\vec{V}_r+(&\langle\vec{\Omega}'\rangle/|\langle\Omega'\rangle|^2)\\&\times\vec{\nabla}_{\perp}[(m_0c^2\gamma')/\omega_{\mathrm{uh}}^2(m_0\gamma)^2].\end{aligned} \tag{62}$$

However, the direct QSA limit has given different results for the potential Ψ', since we have found it either undetermined or equal to zero to second order in q. But going to fourth order in q, we see that this limit gives a nonvanishing result. Thus, one should be cautious about the proper way to handle this limit. Indeed, the tricky fact here is that the expansion should be made up to fourth order in q in order to find a second-order source term, since in the expression for $\langle\vec{V}\rangle$, for example, with Eq. (62), one has to divide by the factor

($\vec{\Omega}'$) that scales at least like second order in q to find the relevant sources terms to second order in order q.

General remark. Here, we keep *a priori* the total slow vorticity $\langle\vec{\Omega}\rangle$ as a function of both $\langle\vec{P}\rangle$ ($\langle\vec{V}\rangle$) and $\langle\vec{B}\rangle$ and not as a function of $\langle\vec{B}\rangle$ only.

3. Perturbative expansion for the nonlinear currents

By performing again the q expansion, we may derive a useful relation for $\langle n_h\vec{V}_h\rangle$ that is valid to second order at least in q. To do this, we use the linearized equation for the hf density n_h, the definition (19) of $\vec{V}_r$ and the relation (25) to obtain

$$\langle n_h{}^{(1)}\vec{V}_h^{(1)}\rangle \approx \langle n\rangle^{(0)}\,\mathrm{curl}(\vec{r}_h^{(1)}\times\vec{V}_h^{(1)})/2-(\vec{\nabla}\langle n\rangle^0\cdot\langle\vec{r}_h{}^1\rangle\vec{V}_h^{(1)}\rangle - \langle n\rangle^{(0)}\vec{V}_r. \tag{63}$$

Using this result (63) for $\langle n_h\vec{V}_h\rangle$, we may express the total slow current $\langle\vec{J}\rangle=\vec{J}_{\mathrm{slow}}+\langle\vec{J}\rangle_{\mathrm{sc}}$ as

$$\langle\vec{J}^{(2)}\rangle/q_e=\langle n\rangle(\langle\vec{V}\rangle-\vec{V}_r)+\vec{J}_m+\vec{C}(\langle n\rangle), \tag{64}$$

$$\vec{J}_m=\mathrm{curl}(\vec{M}). \tag{65}$$

with

$$\vec{M}=\langle n\vec{\mu}\rangle\approx\langle n\rangle\langle\vec{\mu}\rangle \tag{66}$$

$$\vec{C}^{(2)}(\vec{\nabla}n_0)=-(\vec{\nabla}n_0\cdot\langle\vec{r}_h^{(1)})\vec{V}_h^{(1)}\rangle-\vec{\nabla}n_0\times\langle\vec{\mu}^{(2)}\rangle/q_e. \tag{67}$$

Non-QSA limit. The fully nonlinear currents are obtained to the required order in q (up to fourth order here) by using the expressions Eqs. (55a) and (55b) for the currents $\langle\vec{J}_h\rangle$ and $\langle\vec{J}\rangle$.

$\vec{J}$ *slow current*. The hf density n_h is computed to a given-order k in q with the help of $\vec{V}_h{}^{(k)}$, see for example, Eqs. (50)–(51b) for n_h and Eqs. (14b)–(17) for $\vec{V}_h$. Equation (63) is valid to second order in q and may be valid beyond, depending on the structure of the source terms in the continuity equation, for example, for $n_h^{(3)}$.

$\langle\vec{J}\rangle_{\mathrm{sc}}$ *current*. One notes that $\langle\vec{V}\rangle$ has been calculated in various cases (within or not the QSA approximation) considering the ponderomotive force as given. Thus, the components of the currents are computed (self consistently) as far as possible as functions of $\langle n\rangle$, $\langle\vec{B}\rangle$, $\vec{A}_h$, and of the slow source terms such as the ponderomotive fourth-order potential ($\vec{A}_p,\Phi_p$) or such as the term $\langle\gamma_h\vec{V}_h\rangle$ that are coming from the time average of the quadratic product of hf components of lower order.

The self-consistent current is calculated with $\langle\vec{V}\rangle$ given by Eqs. (59) and (60) in the general case. The slow-induced density $\langle n\rangle$ in $\langle\vec{J}\rangle_{\mathrm{sc}}$ will be related to the potential Φ by the Poisson equation (see Sec. III).

QSA limit for the total slow current. In this case, we use the result (62) on $\langle\vec{V}\rangle$ to write the total slow current $\langle\vec{J}\rangle$ in the QSA limit as

$$\langle\vec{J}\rangle/q_e=\langle n\rangle\langle\vec{\Omega}'\rangle\times\vec{\nabla}\Psi'/|\langle\vec{\Omega}\rangle|^2+\vec{J}_m+\vec{C}(\langle n\rangle). \tag{68}$$

Note the compensation of the $\vec{V}_r$ term in $\langle\vec{J}\rangle$, since it occurs in both Eq. (62) for $\langle\vec{J}\rangle_{\mathrm{sc}}$ and in Eq. (63) for $\langle\vec{J}_h\rangle$ but with the opposite sign. This compensation is only partial in the non-QSA case.

Now, we may use the explicit result on the Ψ potential in the QSA, to get the slow currents in the final form

$$\langle\vec{J}\rangle/q_e=\langle n\rangle\langle\vec{\Omega}'\rangle\times\vec{\nabla}_\perp(m_0c^2\gamma')/\omega_{\mathrm{uh}}^2(m_0\gamma)^2+\vec{J}_m+\vec{C}(\langle n\rangle), \tag{69}$$

$$\langle\vec{J}\rangle=\langle\vec{J}_{\mathrm{sc}}\rangle+\vec{J}_m+\vec{C}(\langle n\rangle),$$

$$\langle\vec{J}_{\mathrm{sc}}\rangle=\langle\vec{J}_{p_1}\rangle+\langle\vec{J}_{p_2}\rangle,$$

$$\mu_0\langle\vec{J}_{p_1}\rangle=k_p^2(\langle n\rangle/n_0)\{\vec{\Omega}_p\times\vec{\nabla}_\perp(m_0c^2\gamma')/(m_0\omega_{\mathrm{uh}}\gamma)^2\} \tag{70}$$

$$\mu_0\langle\vec{J}_{p_2}\rangle=k_p^2(\langle n\rangle/n_0\{\langle\vec{\Omega}\rangle\times\vec{\nabla}_\perp(m_0c^2\gamma')/(m_0\omega_{\mathrm{uh}}\gamma)^2\}. \tag{71}$$

The $\langle n_h\vec{V}_h\rangle$ current could also be written directly as

$$\langle n_h^{(1)}\vec{V}_h^{(1)}\rangle=-(\vec{\nabla}n_0\cdot\langle\vec{r}_h^{(1)})\vec{V}_h^{(1)}\rangle-n_0\langle(\vec{\nabla}\cdot\vec{r}_h^{(1)})\vec{V}_h^{(1)}\rangle. \tag{72}$$

Hence, it is zero when $\vec{\nabla}n_0=\vec{0}$ and $\vec{\nabla}\cdot\vec{r}_h^{(1)}=0$. In the relativistic case, the last term could be nonzero since γ is a function of position. When $\vec{J}_{\mathrm{slow}}=\vec{0}$ the total slow current then reads simply

$$\langle\vec{J}\rangle/q_e=\langle n\rangle(\vec{V}_r+\langle\vec{\Omega}'\rangle\times\vec{\nabla}_\perp(m_0c^2\gamma')/\omega_{\mathrm{uh}}^2(m_0\gamma)^2. \tag{73}$$

We shall comment on these currents in the next section.

III. THE VECTORIAL EVOLUTION EQUATIONS

A. General case (arbitrary pump pulses)

The equations of motion for $\vec{P}$ or $\vec{V}$ have to be completed by the Maxwell and the fluid equations. We have derived the various nonlinear currents in Sec. II and we now look at the corresponding evolution equations for the generation of $\langle n\rangle$ and $\langle\vec{B}\rangle$ and for the evolution of the pump vector potential $\vec{A}_h$.

1. The evolution equation for the pump pulse vector potential $\vec{A}_h$

Using the Maxwell equations, we get the propagation equation for $\vec{A}_h$ as

$$D\equiv\partial_{t^2}^2/c^2-\Delta,$$

$$D(\vec{A}_h)=\mu_0\vec{J}=k_p^2(n\vec{P}/\gamma)_h/(n_0q_e)=\mu_0\vec{J}_h^t-\vec{J}_a, \quad (74)$$

$$\vec{J}_h^t=q_e[(n_h\langle\vec{V}\rangle^t+\langle n\rangle\vec{V}_h^t)+(n_h\vec{V}_h)^t],$$

$$\vec{J}_a=\vec{\nabla}\cdot(\vec{A}_h)+\partial_t(\Phi_h)/c^2. \quad (75)$$

In the Lorentz gauge

$$\vec{J}_a=\vec{0}. \quad (76)$$

In the Coulomb gauge $\vec{J}_a=\vec{0}$, but with $\vec{\nabla}\cdot(\vec{A}_h)=0$ and $\Phi_h=0$.

We assume that an envelope approximation for $\vec{A}_h$ is possible, consistent with the multitime and space scalings assumed initially. This approximation is possible even for short pulses if the duration of the pulse τ is long enough as compared to the fast period T_h (a condition necesary to define the fundamental frequency of the pulse itself). The vector potential is written with a fast phase ϕ_h and a slow complex amplitude $\Sigma_{(s)}$. The amplitude $\Sigma_{(s)}$ is itself the product of a slow modulus and of a slow phase ϕ_S, so we get the envelope propagation Eq. (80)

$$\vec{A}_h=\hat{e}_A\Sigma_s e^{-i\varphi_{(h)}(r,z,t)}, \quad (77)$$

$$\varphi_h=\omega t-kz[+\varphi_0(r,z,t)], \quad (78)$$

$$\Sigma_s=|\Sigma_s|e^{-i\varphi_s(r,z,t)},$$

for

$$(\partial_t\Sigma^s/\omega\Sigma^s)\approx\delta'\leqslant 1,$$

and for

$$(\partial_z\Sigma^s/k\Sigma^s)\approx\alpha_{\parallel}\leqslant 1,$$

$$\delta'=1/(\omega\tau)=\delta/(\omega_{\mathrm{pe}}\tau), \quad (79)$$

$$D(\vec{A}_h)=D(\Sigma\hat{e}_A)e^{-i\varphi}+[(\omega^2/c^2-k^2)+2i(k\partial_z+\omega/c^2\partial_t)]\Sigma e^{-i\varphi}\hat{e}_A. \quad (80)$$

A time scale parameter δ' appears in Eq. (79) depending on $\omega_{\mathrm{pe}}\tau$. The envelope Σ evolves on the same time scale than $\langle n\rangle$ for $\omega_{\mathrm{pe}}\tau=1$, faster for $\omega_{\mathrm{pe}}\tau<1$ (the short-pulse case) or slower for $\omega_{\mathrm{pe}}\tau>1$ (the long-pulse case). Similar conditions on the evolution of Σ are found with respect to $\langle\vec{B}\rangle$ by introducing the parameter $\omega_C\tau_{\langle\vec{B}\rangle}$, $\tau_{\langle\vec{B}\rangle}$ being a typical rising time for $\langle\vec{B}\rangle$. The linear dispersion relation of the initially unmagnetized plasma is recovered as

$$D(\vec{A}_h^1)=\mu_0\vec{J}_h^{(1)t} \quad \text{or} \quad \omega^2=k^2c^2+k_p^2c^2. \quad (81)$$

The evolution equation for $\vec{A}_h(\Sigma)$ reads

$$(D+k_p^2+2ik\partial_V)\Sigma\hat{e}_A\equiv G(\Sigma)\hat{e}_A=\mu_0(\vec{J}_h)e^{+i\varphi_h}=(k_p^2\langle n\rangle/n_0\langle\gamma\rangle)\Sigma\hat{e}_A+\vec{R}, \quad (82)$$

$$\partial_z+\partial_t/c\equiv\partial_V. \quad (83)$$

In Eq. (82), the remainder $\vec{R}$ involves the vectorial corrections in the current source terms for the $\vec{A}_h$ envelope evolution.

Now the resolution of $\vec{A}_h$ depends obviously on the degree of resolution chosen for n, $\vec{P}$, and γ. For example, in the plane-wave case the lowest-order (first order in q) approximation gives for tranverse variables: $\vec{P}_h=-q_e\vec{A}_h$ and the right-hand side (rhs) of Eq. (82) yields a scalar coupling in $(n/n_0)\vec{A}_h/\langle\gamma\rangle$ with $\vec{R}=0$. Another presentation of the evolution equations in terms of vorticity could be found in Ref. [7].

Now, we may explain the relevant $\vec{J}_h$ current introduced in Sec. II with Eq. (51) to third order in q around the frequency ω of the pump pulse.

Keeping the possible processes of harmonic generation (to third order in q). It will be important to keep these harmonic currents in particular for the proper treatment of the critical ($\delta=1$) and overdense plasmas ($\delta>1$). Hence, we get

$$\vec{J}_h^{t(3)}(\omega)|_{(q^3,\omega)}/q_e=[n_h^{(1)}(q,\omega)\langle\vec{V}^{(2)}\rangle(q^2)]^t+[\langle n\rangle^{(0)+(2)}\vec{V}_h^{t(1)+(3)}(\omega)]|_{(q^3,\omega)}+\vec{J}_h^{t(3)}(2\omega)|_{(q^3,\omega)}/q_e, \quad (84)$$

$$\vec{J}_h^{t(3)}\{(2\omega)|_{(q^3,\omega)}/q_e=[n_h^{(1)}(q,\omega)+n_h^{(2)}(q^2,2\omega)]\cdot[\vec{V}_h^{(1)}(q,\omega)+\vec{V}_h^{(2)}\times(q^2,2\omega)^t]\}|_{(q^3,\omega)}], \quad (85)$$

$$\vec{J}_h^{t(3)}|_{(q^3,\omega)}/q_e=\{n_0\vec{V}_h^{(3)}+(n_h^{(1)}\langle\vec{V}\rangle^{(2)})+\langle n\rangle^{(2)}\vec{V}_h^{(1)}+(n_h^{(2)}V_h^{(1)}+n_h^{(1)}\vec{V}_h^{(2)})\}|_{(q^3,\omega)}. \quad (86)$$

Here, the current arising at the second harmonic 2ω of the pump frequency contributes to the total current to third order in q through the terms $n_h(2\omega)$ and $\vec{V}_h(2\omega)$, but with an amplitude depending on the scale parameters δ and α.

Neglecting the harmonic terms (to third order in q). In this situation, the current simplifies since we can drop the harmonic contribution (85) to the nonlinear current

$$\vec{J}_h^t|_{(q^3,\omega)}/q\approx n_0\vec{V}_h^{(3)}+(n_h^{(1)}\langle\vec{V}\rangle^{(2)})+\langle n\rangle^{(2)}\vec{V}_h^{(1)}. \quad (87)$$

Using Eq. (16), we get the current and the pump envelope evolution as

$$\mu_0\vec{J}_h^{(3)}=-k_p^2(\langle n\rangle/n_0\langle\gamma\rangle)\vec{A}_n-(k_p^2/i\omega n_0)[\vec{\nabla}\cdot(n_0/\langle\gamma\rangle)\cdot\vec{A}_h]\times\langle\vec{V}\rangle^{(2,t)}\ldots-(k_p^2/\langle\gamma\rangle)\vec{\vec{U}}_{\omega_0\langle\Omega\rangle/m_0(\langle\gamma\rangle}(\vec{A}_h\times\langle\vec{\Omega}\rangle)/[-i\omega(m_0\langle\gamma\rangle)] \quad (88)$$

with $\vec{\nabla}\cdot(\vec{A}_h)=0$,

$$[D(.)+k_p^2+2ik\partial_V]\Sigma\hat{e}_A=k_p^2[(\langle n\rangle/n_0\langle\gamma\rangle]\overleftrightarrow{U}_{\omega,(\langle\Omega\rangle/m_0\langle\gamma\rangle)}\times(\Sigma\hat{e}_A)+\cdots+k_p^2(1/i\omega n_0)\times[\vec{\nabla}\cdot(n_0/\langle\gamma\rangle)\cdot\Sigma\hat{e}_A]\langle\vec{V}\rangle^{(2,t)}). \quad (89)$$

Equation (89) yields the expression of the vectorial correction $\vec{R}$ term of Eq. (82). It shows that the evolution equation for $\vec{A}_h(\Sigma)$ has a vectorial character (independently of the envelope approximation) due to the $\overleftrightarrow{U}$ operator, except in specific cases (such as for pump circular polarization where $\overleftrightarrow{U}$ becomes a scalar). We may see also the nonlinear coupling of $\vec{A}_h(\Sigma)$ with $\langle n\rangle$ and with $\langle\vec{B}\rangle$ through the variables $\langle\vec{\Omega}\rangle$, $\langle\vec{\Omega}_p\rangle$, and $\langle\vec{V}\rangle$.

The velocity term in $\langle\vec{V}\rangle^t$ given by Eq. (60) could bring components nonparallel to the $\vec{A}_h$ direction adding another vectorial character, but in inhomogeneous plasmas only. In the special case of negligible magnetic field and/or vorticity, Eq. (82) describes the coupling between $\vec{A}_h$ and $\langle n\rangle$ only (for $R=0$). It reduces to a scalar equation that has been intensively investigated in the context of relativistic self-focusing (RSF hereafter) studies. The operator D is often further simplified by using the paraxial approximation with the replacement of the D'Alembertien D by the transverse Laplacian operator of diffraction: $D\rightarrow\Delta_\perp$.

Equations (82) or (89) now have to be coupled with the dynamical evolution equations for $\langle n\rangle$ and $\langle B\rangle$.

2. The generation equation for $\langle\vec{B}\rangle$

In order to study the magnetic-field generation, we derive the evolution equation for $\langle\vec{B}\rangle$. We start with the Maxwell-Ampere equation written on the slow scale as

$$\mathrm{curl}(\langle\vec{A}\rangle)=\langle\vec{B}\rangle,$$

$$\mathrm{curl}(\langle\vec{B}\rangle)=\mu_0(\langle\vec{J}\rangle)^t-\partial_{t^2}^2\langle\vec{A}\rangle/c^2. \quad (90)$$

Using the relations (63) for $\langle n_h\vec{V}_h\rangle$, Eq. (64) for the slow current $\langle\vec{J}\rangle$ and for (44) the term $\vec{C}$, we get the evolution equation for $\langle\vec{A}\rangle$ (or $\langle\vec{B}\rangle$) as

$$D(\langle\vec{A}\rangle)=\mu_0[q_e(\langle n\rangle\langle\vec{V}\rangle^t-\vec{V}_r^t)+\vec{C}(\nabla\langle n\rangle)+\mathrm{curl}(\vec{M})]. \quad (91)$$

The generating sources include the current $\vec{J}_m=\mathrm{curl}(\vec{M})$ that is associated with a finite magnetization of the medium.

The time Fourier component of $\langle\vec{A}\rangle$ at the frequency ω_0 reads

$$(-\omega_0^2/c^2-\Delta)\langle\vec{A}\rangle(\omega_0)=\mu_0[q_e\langle n\rangle(\langle\vec{V}\rangle^t-\vec{V}_r^t)+\vec{C}(\nabla\langle n\rangle)+\mathrm{curl}(\vec{M})], \quad (92)$$

$$\langle\vec{V}\rangle^t(\omega_0)\approx[\nabla_\perp(m_0c^2\langle\gamma'\rangle)/(1-\omega_{\mathrm{uh}}^2/\omega_0^2)]\times\langle\vec{\Omega}'\rangle/(\omega_0m_0\langle\gamma\rangle)^2+\vec{D}_1(\omega_0). \quad (62')$$

The $\vec{D}_1$ term in $\langle\vec{V}\rangle^t$ defined with Eq. (62′) represents mainly the non-QSA contribution to $\langle\vec{V}\rangle$, while the leading term is the first one (in the QSA limit).

We see that there are various terms participating to the creation of $\langle\vec{B}\rangle$ in the collisionless plasma case. The magnetization $\vec{M}$ through the magnetization current $\vec{J}_m$, the ponderomotive vector potential $\vec{A}_p$ (connected to the velocity $\vec{V}_r$), and the longitudinal ponderomotive force through the term proportional to $\vec{\nabla}_\perp(\gamma')\times\langle\vec{\Omega}'\rangle$ to quote the main terms.

For a recent review on mechanisms leading to the generation of $\langle\vec{B}\rangle$, see Refs. [7,11]. The velocity $\langle\vec{V}\rangle^t$ contributes to $\langle\vec{A}\rangle$ by both its potential and rotational components and the description is self-consistent if we keep the sources plus induced terms in the nonlinear answers at the same order of expansion. One may specialize further the discussion by investigating the short- and long-pulse cases, by making the QSA approximation or not.

Equation (92) depends on various parameters such as the initial wave polarization, the existence of an external $\vec{B}_0$ magnetic field, the existence of a magnetic moment $\langle\vec{\mu}\rangle$, the inhomogeneity of n_0, the conservation of vorticity or not, and so on. We shall investigate in the following a few simple cases only.

3. QSA limit for magnetic field generation

In the QSA limit, the velocity $\langle\vec{V}\rangle$ reduces to Eq. (62) with the coefficient D_1 in (93) going partially to zero with ω_0. Thus, the remaining source terms for $\langle\vec{B}\rangle$ generation are now found as

$$\Delta(\langle\vec{A}\rangle)=\mathrm{curl}\langle\vec{B}\rangle=\mu_0\{q_e\langle n\rangle[\langle\vec{\Omega}'\rangle\times\vec{\nabla}_\perp(m_0c^2\gamma')/\omega_{\mathrm{uh}}^2(m_0\gamma)^2]\}+\cdots+\mu_0\,\mathrm{curl}(\vec{M})+\mu_0q_e\vec{C}(\nabla\langle n\rangle) \quad (93)$$

for

$$\langle n_h\vec{V}_h\rangle\neq\vec{0}:$$

$$\mathrm{curl}[\vec{\omega}_c-(\mu_0q_e/m_0)\vec{M}]=-k_p^2\{c^2(\langle n\rangle/n_0)[\langle\vec{\Omega}'\rangle/(m_0)\times\vec{\nabla}_\perp(\gamma')/(\omega_{\mathrm{uh}}\gamma)^2]+\vec{C}/n_0\}, \quad (94)$$

for

$$\langle n_h\vec{V}_h\rangle=\vec{0}:$$

$$\mathrm{curl}(\vec{\omega}_c)=-k_{\mathrm{po}}^2\{c^2(\langle n\rangle/n_0)[\vec{V}_r+\langle\vec{\Omega}'\rangle/(m_0)\times\vec{\nabla}_\perp(\gamma')/(\omega_{\mathrm{uh}}\gamma)^2]+\vec{C}/n_0\}. \quad (95)$$

Analysis of the source currents. We may now identify the main sources of magnetic-field generation in the underdense and collisionless plasma from Eqs. (94) and (95). There are given by (i) the finite-magnetization $\vec{M}$ that occurs in the inverse Faraday effect in the case of pulses with elliptical polarization of their electric fields [12,13]; (ii) the inhomogeneity of n_0 through the term $\vec{C}$; and (iii) the self-consistent slow nonlinear current that is split into the two contributions $\vec{J}_{p1}$ and $\vec{J}_{p2}$ [see Eqs. (70) and (71)] that are involving the possible finite $\vec{A}_p$ in ω_{uh} and through Ω' and $\vec{V}_r$ and the creation of $\langle\vec{B}\rangle$ by the self-consistent current source $\langle J\rangle_{p1}$ corresponding to an incident wave of finite-spatial extension. This last contribution is coming from the longitudinal ponderomotive force $\vec{F}_{p,1}$ through the self-consistent term in the equation of motion for $\langle\vec{V}\rangle$ (i.e., by the $\vec{F}_{p,1}\times\langle\vec{B}\rangle$ term).

Magnetization current $\vec{J}_m$. This current $\vec{J}_m$ arises from a finite-magnetization $\vec{M}$ and leads to the so-called inverse Faraday effect (IFE) for circularly or elliptically polarized pulses. This effect vanishes for both homogeneous density and pump electromagnetic field. The $\vec{C}$ term comes together with $\vec{J}_m$ in $\langle\vec{J}_h\rangle$. It brings a finite contribution in the case of inhomogeneous density and also in case of a pump with an electrostatic component (through a source or an induced density n_h). For a bounded plasma with an (unphysical) homogeneous density, there might be a residual component of $\langle\vec{B}\rangle$ coming from a surface effect [8b]. In laser-produced plasmas, the density profile reflects the laser intensity profile characteristics with large inhomogeneities of n_0 and of $\vec{E}_h$ ($\vec{B}_h$) in a transverse direction with respect to propagation for focused pulses in underdense plasmas. Thus the term $\vec{C}$ could be large. The density inhomogeneity is also enhanced by the ponderomotive force in the case of pulse chaneling, since the total density is to second order in q: $\langle n\rangle_{\rm tot}-n_0=\langle n\rangle(q^2)+\cdots$.

In overdense plasmas, additional effects could come from large axial inhomogeneities of both pump and mean density (through skin effects [5(b)]).

The self-consistent current $\vec{J}_{\rm sc}$. This current $\vec{J}_{\rm sc}$ is split into the two currents $\vec{J}_{p1}$ and $\vec{J}_{p2}$ that are of ponderomotive origin (index p). There are nonlinear self-consistent terms coming from the $\langle n\rangle\langle\vec{V}\rangle$ current.

The first term $\vec{J}_{p1}$ is proportional to the vectorial product of the ponderomotive magnetic-field $\vec{B}_p=\mathrm{curl}(\vec{A}_p)$ with the longitudinal ponderomotive force $\vec{F}_{p,1}$ or $\vec{\nabla}_\perp(m_0c^2\gamma')$ as seen with Eq. (70). We have seen also that $\vec{A}_p$ is connected with the transversal part of the fluid ponderomotive force $\vec{F}_{p,t}$.

The creation of magnetic field could occur also through the current $\vec{J}_{p2}$, which is again of a ponderomotive origin. This term is self consistent in the sense that it includes directly the $\langle\vec{B}\rangle$ field to be computed through the vorticity $\langle\Omega\rangle$ in the vectorial product $\langle\vec{\Omega}\rangle\times\vec{\nabla}_\perp(m_0c^2\gamma')$. This last contribution could be often the strongest source of $\langle\vec{B}\rangle$ in practice (see application below). Note that in a q expansion, the relevant current is obtained to fourth order in q, but the induced magnetic field could be found with an amplitude at lower order in q.

Remark. In Ref. [14], it was noted that the $\langle\vec{B}\rangle$ field could have two origins, one due to magnetization (IFE) and the other due to a ponderomotive origin but in this context, there is first creation of an induced n_h wakefield (to third order in q) and then of a $\langle\vec{B}\rangle$ field proportional to fourth order in q by the current $\langle\vec{J}_h^{(4)}\rangle=q_e\langle n_h^{(3)}\vec{V}_h^{(1)}\rangle$ (see also Ref. [15]). Using our approach, we shall find a ponderomotive induced $\langle\vec{B}\rangle$ field proportional to q with the self-consistent ponderomotive generation mechanism for circularly polarized pulses, while the $\langle\vec{B}\rangle$ field predicted by the usual IFE scales as q^2 (see, for example, Refs. [8b, 16]).

4. The generation equation for the slow density $\langle n\rangle$

The induced density is related to the potential ϕ by the Poisson equation.

$$\langle\vec{E}\rangle^l=-\vec{\nabla}\langle\phi\rangle,\epsilon_0\partial_t\langle\vec{E}\rangle^l+\langle\vec{J}\rangle^l=0, \tag{96}$$

$$\langle n\rangle=n_0[1-k_p^{-2}\Delta(\langle\phi\rangle)/\phi_c], \tag{97}$$

$$\phi_c=m_0c^2/(+q_e),$$

$$\langle n\rangle=n_0\{1-k_p^{-2}\Delta(\Psi'/q_e\phi_c-\langle\gamma'\rangle)\}. \tag{98}$$

Where the definition (35) for Ψ' has been used

$$\langle n\rangle(\omega_0)=n_0\{1-k_p^{-2}[\nabla_\parallel\nabla_\parallel(\Psi')+\nabla_\perp\nabla_\perp(\Psi')]/q_e\phi_c-[(\Delta_\parallel+\Delta_\perp)\langle\gamma'\rangle]\}. \tag{99}$$

In Eq. (99), one has to substitute the expressions of $\vec{\nabla}_\parallel(\Psi^{(')})$ and $\vec{\nabla}_\perp(\Psi^{(')})$ from Eqs. (56),(57) in the non-QSA or by, Eqs. (61a),(61b) in the QSA limit.

We see that $\langle n\rangle$ may be modified by the low-frequency magnetic-field $\langle\vec{B}\rangle$ because of their intricate coupling through $\vec{V}_r$ and $\vec{\Omega},\vec{\Omega}'$ entering in Ψ' and γ'. We can now compute the density perturbation to relevant second or fourth order in q.

$B=0$ limit. In the case of negligible $\langle\vec{B}\rangle$, the equation for $\langle n\rangle$ reduces to Eq. (98) without the prime index in Ψ and γ.

$$\Delta\Psi'=\vec{\nabla}\cdot[a(\langle n_h\vec{V}_h\rangle+\langle n\rangle(-\vec{V}_1))]+\vec{\nabla}\cdot[aa_2\nabla(m_0c^2\langle\gamma\rangle)],$$

$$a_2=-i\epsilon_0\omega_0/e^2,\quad a=1/[a_2(1-\omega_{\rm pe}^2/(\omega_0^2\langle\gamma\rangle)]. \tag{100}$$

The QSA limit is easily recovered using Eqs. (61a), (61b) for ψ' and in the zero magnetic-field case, one recovers the well-known result

$$\lim_{\omega_0\to 0}\langle n\rangle=n_0\{1+k_p^{-2}\Delta\langle\gamma\rangle\}. \tag{101}$$

Finite $\langle\vec{B}\rangle$ *limit*. For the *non*-QSA *limit*; see the dynamical equation for $\langle n\rangle$ below. For the QSA *limit*, the $\langle n\rangle$ generation equation reads (to fourth order in q)

$$\lim_{\omega_0\to 0}\langle n\rangle/n_0=\{1-k_p^{-2}[\Delta_\perp(\Psi'/q_e\phi_c)-\Delta\langle\gamma'\rangle]\}$$
$$=1+k_p^{-2}\Delta_\parallel\langle\gamma'\rangle-\cdots-k_p^{-2}\langle\Omega'\rangle^2/\omega_{\rm uh}^2(m_0\langle\gamma\rangle)^2-1)\Delta_\perp(\langle\gamma'\rangle)+k_p^{-2}\nabla_\perp(\langle\gamma'\rangle)\cdot\nabla_\perp\{[\langle\Omega'\rangle^2/\omega_{\rm uh}^2(m_0\langle\gamma'\rangle)^2]\}. \qquad (102)$$

With γ' given by Eq. (22) and Ω' by Eq. (34b) in Eqs. (99)–(102). Simplifications may be done by expanding $\langle P\rangle$ to the relevant q order in γ' and Ω'. Note also that $\langle n\rangle$ is not modified by the low-frequency magnetic field to second order in q, since in the second line of the rhs of Eq. (102), we have for

$$\langle\Omega'\rangle/m_0\approx\omega_c,$$

$$[\langle\Omega'\rangle^2/\omega_{\rm uh}^2(m_0\langle\gamma\rangle)^2-1]\Delta_\perp(\gamma')\approx-\Delta_\perp(\gamma')(\omega_{\rm pe}^2/\gamma)/[(\omega_{\rm pe}^2/\gamma)+\omega_c^2/\gamma^2]\to\cdots-\Delta_\perp[(\gamma')\times(1+\vartheta(q^4)].$$

The nonlinear couplings between $\vec{A}_h$, $\langle n\rangle$, and $\langle\vec{B}\rangle$ through, respectively, the Eqs. (89)-(94)-(98) are starting at least with fourth order.

Dynamical equation for slow density $\langle n\rangle$. Another way to derive the equation for $\langle n\rangle$ generation is obtained directly in the time domain by starting from the basic equations of Sec. II. By taking the time derivative of the continuity equation for $\langle n\rangle$ and substituting the value of $\langle\vec{V}\rangle$ from the equation of motion, we can write

$$[\partial_{t_s^2}+\omega_{\rm pe}^2(\langle n\rangle)/\langle\gamma\rangle]\langle n\rangle=W_1+W_2+W_3-(q_e/m_0)\vec{\nabla}\cdot(n_0\vec{E}_p/\langle\gamma\rangle), \qquad (102')$$

$$\langle\vec{V}'\rangle\equiv\langle\vec{V}\rangle+\langle\gamma_h\vec{V}_h\rangle/\langle\gamma\rangle,\vec{E}_p\equiv-\vec{\nabla}(\Phi_p),$$

$$W_1(\partial_t)=-\vec{\nabla}\cdot[\partial_t(\langle n\rangle)\langle\vec{V}\rangle+\langle n\rangle(\langle\vec{V}\rangle\partial_t\langle\gamma\rangle+\partial_t\langle\gamma_h\vec{V}_h\rangle)/\langle\gamma\rangle]$$

$$W_2(\nabla)=-(q_e/m_0)\vec{E}\cdot\vec{\nabla}(n_0/\langle\gamma\rangle)-\vec{\nabla}\cdot[n_0(\langle\vec{V}'\rangle\cdot\vec{\nabla})\times(\langle\gamma\rangle\langle\vec{V}'\rangle)/\langle\gamma\rangle)],$$

$$W_3(\langle\vec{B}\rangle)=-(q_e/m_0)\vec{\nabla}\cdot[n_0/\langle\gamma\rangle(\langle\vec{V}'\rangle\times(\langle\vec{B}\rangle+\vec{B}_p)].$$

In the limit $\langle\vec{B}\rangle=\vec{0}$, the remaining dynamical equation for $\langle n\rangle$ is obtained to a given order m as

$$[\partial_{t^2}+\omega_{\rm pe}^2(\langle n\rangle)/\langle\gamma\rangle]\langle n\rangle^{(m)}=S^{(m)},$$

for example to second order in q

$$S^{(2)}=\sum_i H_i^{(2)}-(q_e/m_0)\vec{\nabla}\cdot(n_0\vec{E}_p^{(2)}/\langle\gamma\rangle^{(0)}), \qquad (102'')$$

$$\sum_i H_i^{(2)}=-\vec{\nabla}\cdot[\partial_t(n_0)\langle\vec{V}^{(')}\rangle^{(2)}]-(q_e/m_0)\langle\vec{E}\rangle_{\rm sc}^2\cdot\vec{\nabla}(n_0/\langle\gamma\rangle^{(0)})$$

$$[\partial_{t^2}+\omega_{\rm pe}^2(n_0)/\langle\gamma\rangle]\langle n\rangle^{(2)}=-\{q_e/m_0)[\nabla\cdot(n_0\vec{E}_p^{(2)}/\langle\gamma\rangle^{(0)}]+\langle\vec{E}\rangle_{\rm sc}^{(2)}\cdot\vec{\nabla}(n_0/\langle\gamma\rangle^{(0)})\},$$

for an homogeneous density n_0

$$[\partial_{t^2}+\omega_{\rm pe}^2(n_0)/\langle\gamma\rangle^{(0)}]\langle n\rangle^{(2)}\approx(n_0\Delta\Phi_p^{(2)}/m_0\langle\gamma\rangle^{(0)}). \qquad (102''')$$

The QSA limit of Eq. (102‴) again gives the result (101) for $\Phi_p^{(2)}$ that is equal to $m_0c^2[\gamma^{(2)}-1]$.

5. Closure equation for the slow vorticity

The zero vorticity case. When the total vorticity is initially zero, it is conserved from Eq. (6) and hence, the slow magnetic field is given to any order in q by the simple formula

$$q_e\langle\vec{B}\rangle=-\mathrm{curl}(\langle\vec{P}\rangle\to\langle\vec{B}\rangle=\vec{B}_p=\mathrm{curl}(\vec{A}_p). \qquad (103)$$

In this case, it is convenient to work with a closed nonlinear equation, for example, on P that is valid to all orders in q [7,17].

The nonzero vorticity case. If we consider rather that the initial conditions for the pump onset are such (sudden versus adiabatic turning on of the pump) that they bring vorticity perturbations, we need to solve another equation for the (slow) vorticity.

For example, we may write

$$\langle\vec{\Omega}\rangle/m_0=(\langle\omega_c\rangle+\mathrm{curl}\langle\gamma\rangle\vec{V}_r)[+\mathrm{curl}(\delta\vec{P}_r)/m_0]. \qquad (104)$$

If to second order q we may neglect $\delta\vec{P}_r=\langle\vec{P}\rangle-m_0\langle\gamma\rangle\vec{V}_r$, we are left with

$$\langle\vec{\Omega}\rangle/m_0=(\langle\omega_c\rangle+\mathrm{curl}(\langle\gamma\rangle\vec{V}_r). \qquad (105)$$

A simpler assumption is merely to take the vorticity $\langle\vec{\Omega}\rangle/m_0$ to be equal to $q_e\langle\vec{B}\rangle$ to lowest consistent (second order in q).

Closure problem [7,9]. In fact, taking the time average of Eq. (1) for the vorticity leads to a compatibility condition that implies a (nonlinear) differential equation (on the spatial variable $\vec{r}$) on $\langle\vec{\Omega}\rangle$ itself when $\langle\vec{P}\rangle$ is expressed as a function of $\langle\vec{B}\rangle$ and of $\langle\vec{\Omega}\rangle$. This equation could be truncated by expanding $\langle\vec{P}\rangle$ in powers of q and provides a corresponding relation for $\langle\vec{\Omega}\rangle$ at a given order.

Generally, we may write this compatibility condition on $\langle\vec{\Omega}\rangle$ as

$$\langle\vec{\Omega}\rangle/m_0-\vec{\omega}_c=\mathrm{curl}\langle\vec{\mathrm{P}}\rangle/m_0=\mathrm{curl}(\langle\vec{\mathrm{V}}\rangle\langle\gamma\rangle+\langle\gamma_h\vec{\mathrm{V}}_h\rangle). \tag{106}$$

In the QSA limit, the compatibility condition is simplified into

$$\langle\vec{\Omega}\rangle/m_0-\vec{\omega}_c=\mathrm{curl}(\langle\gamma_h\vec{\mathrm{V}}_h\rangle)+\mathrm{curl}(\gamma(\vec{\mathrm{V}}_r+c^2\langle\vec{\Omega}'\rangle/m_0\gamma)\times\vec{\nabla}_\perp(\gamma')/\ldots(\omega_{\mathrm{pe}}^2+(\Omega'/m_0)^2/\gamma). \tag{107}$$

To be self consistent, one should add, in fact the evolution equation for $\langle\vec{\mathrm{P}}\rangle$, thus also including the dependence in $\langle\vec{\mathrm{P}}\rangle$ of $\langle\gamma\rangle(\langle\vec{\mathrm{P}}\rangle)$ and of $\langle\vec{\Omega}\rangle(\langle\vec{\mathrm{B}}\rangle,\langle\vec{\mathrm{P}}\rangle)$. So we have, in general, *four coupled nonlinear equations for* $\vec{\mathrm{A}}_h$, $\langle n\rangle$, $\langle\vec{\mathrm{B}}\rangle$, and $\langle\vec{\mathrm{P}}\rangle$ or $\langle\vec{\Omega}\rangle$. But, if one keeps only the source terms ($\langle\vec{\mathrm{J}}_h\rangle$) neglecting the induced ones ($\langle\vec{\mathrm{J}}\rangle_{\mathrm{sc}}$) in a non-self-consistent approach (setting $\langle\vec{\mathrm{V}}\rangle=\vec{0}$ without reliable justification), as it is often done, one is left with three coupled closed differential equations only for $\vec{\mathrm{A}}_h$, $\langle n\rangle$, and $\langle\vec{\mathrm{B}}\rangle$.

B. Special case of circularly polarized pulses

Solution of the equation of motion

In the case of a circularly polarized transverse pulse with a wave vector directed along the z axis the self-generated magnetic field lies also along z (to lowest order in second order q) assuming axisymmetry. The symmetry of the particles trajectories allows the system to remain "integrable" because the helicity (combination of the translation and rotation motions) is conserved. This case of a transverse circular wave propagating along $\vec{\mathrm{B}}$ is well known, except that here $\langle\vec{\mathrm{B}}\rangle$ is an induced nonlinear function of the pump field. The situation with an external magnetic field $\vec{\mathrm{B}}_0$ has been studied in detail [18], where explicit results for trajectories of electrons in terms of elliptic functions were derived. The configuration with circular pulses has been studied in the context of autoresonance [19] useful for particle acceleration and also for electron cyclotron heating, when the phase-matching condition is realized: ($\omega_c/\gamma\approx\omega$). However, the studies are generally restricted to a single particle in given waves, neglecting the collective and self-induced effects on the pump wave (medium polarization effects). Staying far away from the resonance condition ($\omega_c/\gamma\omega\ll1$), since the induced magnetic-field $\langle\vec{\mathrm{B}}\rangle$ will be found smaller than the magnetic-field $\vec{\mathrm{B}}_h$ of the incident wave, we try rather to keep the self-consistent collective effects.

For the circular polarization case, the velocity $\vec{\mathrm{V}}_h$ is parallel to the magnetic-field $\vec{\mathrm{B}}_h$ of the wave, thus the nonlinear term $\vec{\mathrm{V}}_h\times\vec{\mathrm{B}}_h$ in the Lorentz force vanishes at least until third order in q. Second, the Lorentz factor γ is a constant in time, to second order in q.

With these simplifications, the following results are obtained, after choosing the incoming pump vector potential expression:

$$\vec{\mathrm{A}}_{\perp h}=\vec{\mathrm{A}}_0(\epsilon\vec{r},\epsilon t)\{\sin[\varphi(\vec{\mathrm{r}},t)\hat{e}_x+\lambda\cos\varphi(\vec{\mathrm{r}},t)\hat{e}_y]\} \tag{108}$$

$$\varphi(\vec{\mathrm{r}},t)=kz-\omega t+\varphi_0(\vec{\mathrm{r}},t),$$

$\lambda=\pm1$ (right or left polarization).
The Coulomb gauge condition imposes

$$\partial_x[A_0(\epsilon\vec{\mathrm{r}},\epsilon t)\sin(\varphi)]+\partial_y[A_0(\varepsilon\vec{\mathrm{r}},\varepsilon t)\cos(\varphi)]=0, \tag{109}$$

$$\overleftrightarrow{\mathrm{U}}\approx\eta'(|\langle\Omega'\rangle|/m_0)\overleftrightarrow{I}_d, \tag{110}$$

$$\eta^{(\prime)}=1/(1-\lambda|\langle\vec{\Omega}'\rangle|/m_0\omega\langle\gamma\rangle). \tag{111}$$

The $\overleftrightarrow{\mathrm{U}}$ operator reduces here to a scalar, yielding a scalar equation for $\vec{\mathrm{A}}_h(\Sigma)$ instead of the vectorial contribution in $(\vec{\Sigma}\times\Omega)$ for arbitrary pulse polarization.

With the choice of the simplest vorticity closure assuming that Ω' reduces to $m_0\omega_c$, we have

$$\eta=1/(1-\lambda\omega_0/\omega\gamma). \tag{112}$$

An approximate solution is readily obtained by solving for the velocity using a q perturbation.

$$\vec{\mathrm{E}}_\perp=\vec{\mathrm{E}}_0(\epsilon\vec{\mathrm{r}},\epsilon t)\{\cos[\varphi(\vec{\mathrm{r}},t)]\hat{e}_x+\lambda\sin[\varphi(\vec{\mathrm{r}},t)]\hat{e}_y\},$$
$$E_0=(A_0/\omega), \tag{113}$$

$$\vec{\mathrm{V}}_h^{(1+3)}(\omega)\approx(cq\eta'/\gamma)[-(\sin\varphi)\hat{e}_x+\lambda\cos(\varphi)\hat{e}_y], \tag{114}$$

$$n_h^{(1+3)}(\omega)\approx-\vec{\nabla}\cdot(n_0\vec{\mathrm{r}}_h^{(1+3)}+\langle n\rangle^{(2)}\vec{\mathrm{r}}_h^{(1)})\approx\vec{\nabla}_\perp(n_0\eta'/\gamma)(cq/\omega)+\vec{\nabla}\cdot(\langle n\rangle^{(2)}\vec{\mathrm{r}}_h^{(1)}), \tag{115}$$

Here $\vec{\mathrm{F}}_{p,p}$ reduces to (since $\vec{\mathrm{V}}_h^1\times\vec{\mathrm{B}}_h^{(1)}=\vec{0}$)

$$\vec{\mathrm{F}}_p=-m_0\langle(\vec{\mathrm{V}}_h^1\cdot\vec{\nabla})[(\gamma)\vec{\mathrm{V}}_h^1\rangle], \tag{116}$$

$$\vec{\mathrm{F}}_p=-(m_0c^2/2)(\eta'q/\gamma)\{\vec{\nabla}_\perp(\eta'q)+\lambda\eta'q(-\partial_y\varphi_0\hat{e}_x+\partial_x\varphi_0\hat{e}_y)\}, \tag{117}$$

by using the gauge condition: $\vec{\nabla}\cdot(\vec{q}_\perp e^{i\varphi})=0$, we get conditions on the slow phase φ_0 as

$$\partial_y\varphi_0=-\partial_xE_0/E_0,\quad \partial_x\varphi_0=\partial_yE_0/E_0,$$

$$\vec{\mathrm{F}}_p=-(m_0c^2)(\eta'q/2\gamma)\{\vec{\nabla}_\perp(\eta'q)+\lambda\eta'\vec{\nabla}_\perp(q)\}. \tag{118}$$

An alternative but similar form for $\vec{\mathrm{F}}_p$ is also

$$\vec{\mathrm{F}}_p = -(m_0 c^2)\vec{\nabla}_\perp(\gamma) + \langle \vec{\mathrm{V}}_h \times \vec{\Omega}_{\mathrm{h}} \rangle, \tag{119}$$

$$\langle \vec{\mathrm{V}}_h \times \vec{\Omega}_h \rangle \approx -\vec{\mathrm{V}}_r \times \langle \vec{\Omega}' \rangle + \vec{\nabla} \cdot (\langle \vec{\mu} \rangle \cdot \langle \vec{\Omega}' \rangle),$$

using Eq. (19) for $\vec{\mathrm{V}}_r$, with $\langle \vec{\Omega}' \rangle \| \langle \vec{\Omega} \rangle \| \vec{z}$

$$\vec{\mathrm{V}}_r = -c^2(\eta' q/2\omega\gamma) \times \begin{Bmatrix} \lambda \partial_y(\eta' q/\gamma) + (\eta'/\gamma)\partial_y(q) \\ -\lambda \partial_x(\eta' q/\gamma) - (\eta'/\gamma)\partial_x(q) \end{Bmatrix} \times \begin{Bmatrix} \hat{e}_x \\ \hat{e}_y \end{Bmatrix}, \tag{120}$$

$$-m_0\Big(\vec{\mathrm{V}}_r \times \langle \vec{\Omega}' \rangle = -m_0 c^2 (\lambda \langle \Omega' \rangle \eta' q/2\omega\gamma') \times \begin{Bmatrix} \partial_x(\eta' q/\gamma) + \lambda(\eta'/\gamma)\partial_x(q) \\ \partial_y(\eta' q/\gamma) + \lambda(\eta'/\gamma)\partial_y(q) \end{Bmatrix} \times \begin{Bmatrix} \hat{e}_x \\ \hat{e}_y \end{Bmatrix}. \tag{121}$$

The two equivalent forms (116) and (119) for $\vec{\mathrm{F}}_p$, allow us to double check the explicit result, Eq. (118).

Thus, the general evolution equations for $\vec{\mathrm{A}}$, $\langle n \rangle$, and $\langle \vec{\mathrm{B}} \rangle$ in the case of circularly polarized pulses are the following:

$$[D(\cdot) + 2ik\partial_V]\Sigma \hat{e}_A = k_{\mathrm{po}}^2\{(\eta' \langle n \rangle / n_0 \gamma - 1)\Sigma \hat{e}_A + [\vec{\nabla}(\langle n \rangle_{(0)}/\gamma) \cdot \Sigma \hat{e}_A]\langle \vec{\mathrm{V}} \rangle^{(2,t)}/i\omega n_0\}, \tag{122}$$

$$\mathrm{curl}[\vec{\omega}_c - (\mu_0 q_e/m_0)\vec{\mathrm{M}}] = (-)k_{\mathrm{po}}^2\{c^2(\langle n \rangle/n_0)[(\langle \vec{\Omega}' \rangle/m_0) \times \vec{\nabla}_\perp(\gamma')/(\omega_{\mathrm{uh}}\gamma)^2] + \vec{C}/n_0\}, \tag{123}$$

$$\nabla_\| \Psi' \approx a \nabla_\| (m_0 c^2 \langle \gamma' \rangle / (1 - \omega_{\mathrm{pe}}^2/(\omega_0^2 \langle \gamma \rangle))], \tag{124a}$$

$$\nabla_\| \Psi' = 0 \quad \text{for} \quad \partial_z \gamma' = 0,$$

$$\nabla_\perp \Psi' \approx b[(\vec{\mathrm{V}}_r^t \times \langle \vec{\Omega} \rangle)_\perp / (i\omega_0 m_0 \langle \gamma \rangle) + \ldots \langle n \rangle i/(1-\Delta)(V_0 - V_1)_\perp^t \times \langle \vec{\Omega}' \rangle / (\omega_0 m_0 \langle \gamma \rangle)] + \ \ldots \nabla_\perp (m_0 c^2 \langle \gamma' \rangle) \times (1-\Delta)/(1 - \omega_{\mathrm{uh}}^2/\omega_0^2), \tag{124b}$$

$$\langle n \rangle = n_0(1 + k_p^{-2}\Delta_{(\perp)}\phi), \tag{125}$$

$$q_e \phi/(-\phi_c) = \Psi'/(-\phi_c) - (\gamma' - 1),$$

$$\nabla_\perp[q_e \phi/(-\phi_c)] = B(\omega_0) + \nabla_\perp(\gamma')(\omega_{\mathrm{peo}}^2/\gamma[\omega_0^2 - \omega_{\mathrm{uh}}^2)]. \tag{126}$$

$\langle \vec{\mathrm{V}} \rangle^{(2)}$ in Eq. (122) is still given by Eqs. (60) and by Eq. (62) in the QSA limit.

One notes that Eq. (125) relating $\langle n \rangle$ and the potential is already a quasistatic equation (even if the potential ϕ remains a function of the slow-frequency ω_0). We have to expand explicitly the equations in q and we have still to deal with the closure problem case since $\langle \vec{\mathrm{P}} \rangle$ enters (to at least fourth order) in γ and γ' and (to at least second order) in the expressions of $\vec{\Omega}$ and $\vec{\Omega}'$.

IV. NUMERICAL SOLUTION OF A SIMPLIFIED SYSTEM IN THE CASE OF CIRCULAR POLARIZATION

We solve numerically the relevant evolution equations imposing a simple compatibility closure condition for the averaged vorticity in this section on a simple case.

A. Physical motivation

In the process of RSF that is described by the two coupled scalar Eqs. (122) and (125) for $\vec{A}_h$ and $\langle n \rangle$, the self-generated magnetic field could play an important role. More precisely, we have shown theoretically [7] the simultaneous occurrence of RSF and of electron confinement within the light beam due to $\langle \vec{\mathrm{B}} \rangle$, a situation leading to an intense electron photon interaction and to possible relativistic magnetic guiding of light. While it has been shown numerically [3,4] that relativistic filamentation of light could be prevented by the $\langle \vec{\mathrm{B}} \rangle$ field. In effect, $\langle \vec{\mathrm{B}} \rangle$ induces a coalescence or a merging of otherwise diverging filaments towards the central part of the light beam leading to magnetic guiding of light.

By undertaking a numerical simulation, we wish to emphasize these physical arguments as an application. Also, we can predict *a priori* a lowering of the threshold power $\vec{\mathrm{P}}^*$ for the RSF process together with a modification of the cavitation threshold (cavitation is defined by the locally spatial vanishing of the total density) in presence of the $\langle \vec{\mathrm{B}} \rangle$ field.

B. The simplified system

We are restricted here to the homogeneous plasma case only. We start from our general set of nonlinear coupled equations in the case of circular polarization as given in Sec. IIIB. We add simplifications by considering the QSA limit and in the case of an initially homogeneous plasma. We are restricted again to axisymmetric solutions in this application. In this case, we are left with a scalar equation for the envelope Σ as

$$[D(.) + 2ik\partial_V]\Sigma \approx k_{\mathrm{po}}^2\{(\eta' \langle n \rangle / n_0 \gamma) - 1\}\Sigma. \tag{127}$$

Using the paraxial approximation $[D(.) \to \Delta_\perp]$, we take the stationary limit for Σ. It is an additional hypothesis that differs from the QSA limit, see the parameter δ', that is, we set $\partial_v \approx 0$. The equation for Σ becomes simpler,

$$\Delta_{\perp(rr)}(\Sigma) \approx k_{\mathrm{po}}^2\{(\eta' \langle n \rangle / n_0 \gamma - 1)\}\Sigma. \tag{128}$$

The other remaining equations are on the slow gyrofrequency ω_c and on the potential ϕ.

In cylindrical geometry $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$ with variables function of r only, $\partial_\theta = 0$,

$$\vec{\omega}_c = \omega_c(r)\hat{e}_z, \quad \mathrm{curl}(\vec{\omega}_c) = (1/r)\partial_r(r\omega_c)\vec{e}_\theta,$$

$$\text{curl}[\vec{\omega}-(\mu_0 q_e/m_0)\vec{\text{M}}]\cdot\hat{e}_\theta=\vec{\text{S}}(\omega_c)\cdot\hat{e}_\theta, \quad (129)$$

$$\vec{\text{S}}(\omega_c)=k_{\text{po}}^2\{[\langle\vec{\Omega}'\rangle(\omega_c)/m_0]\times[\vec{\nabla}_\perp(\gamma')]/\gamma[\omega_{\text{pe}}^2+(\langle\vec{\Omega}'\rangle/m_0)^2/\gamma]\}. \quad (130)$$

The magnetization $\vec{\text{M}}$ and velocity $\vec{\text{V}}_r$ are given by the expressions

$$\vec{\text{M}}=\epsilon(-m_0\omega/2\mu_0 e)(\delta^2 q^2\eta'^2)/2\gamma^2\hat{e}_z, \quad (131)$$

$$\vec{\text{V}}_r\cdot\hat{e}_\theta=-(c^2 q\eta')/\omega\gamma)\{\lambda\partial_r(\eta' q/\gamma)+(\eta'/\gamma)\partial_r(q)\}. \quad (132)$$

The closure problem for vorticity is solved by setting to second order in q:$\langle\Omega'\rangle=\langle\Omega\rangle=m_0\omega_c$ (133).

It consists of neglecting the $\vec{\text{A}}_p$ or $\vec{\text{V}}_r$ contribution to the slow-magnetic field since

$$\omega_c \gg (2)\omega_{\text{cp}}\approx\text{curl}(V_r):\langle\vec{\Omega}'\rangle^{(2)}/m_0\approx\omega_c,$$

$$\eta'\approx 1\pm(\omega_c/\omega\gamma), \quad (133)$$

$\langle P\rangle^{(2)}\sim\gamma V_r\approx 0$ (but $\langle\vec{\text{V}}\rangle$ and $\vec{\text{V}}_r$ are both $\neq\vec{0}$ prior to this final assumption).
Then, the magnetization factor η' reduces to

$$\eta'\approx 1+\lambda(\omega_c/\omega\gamma). \quad (134)$$

The equation for the potential reads

$$\vec{\nabla}_{(\perp)}(\phi)/(-\phi_c)\approx-\vec{\nabla}_{(\perp)}(\gamma')/(1+(\vec{\Omega}'/\omega_{\text{pe}}m_0)^2(/\gamma), \quad (135\text{a})$$

or by using the q expansion

$$\vec{\nabla}_{(\perp)}(\phi)/(-\phi_c)(q^4)\approx\vec{\nabla}_{(\perp)}(\gamma')(q^4)+(\vec{\Omega}'^2/\gamma)(q^2)\times\vec{\nabla}_{(\perp)}(\gamma^{(')})(q^2)/(\omega_{\text{pe}}m_0)^2. \quad (135\text{b})$$

For the Lorentz factors, we have

$$\gamma|_{q^4}=[1+(\eta' q)^2]^{1/2}\approx[1+(\eta'|_{q^2})q^2]^{1/2}, \quad (136\text{a})$$

$$\gamma|_{q^4}\approx\{1+q^2+2\lambda q^2[\omega_c/\omega(\gamma)]\}^{1/2}, \quad (136\text{b})$$

$$\gamma^{(')}|_{q^4}=[1+(\eta' q)^2+2\langle\vec{u}\rangle^{(2)}\cdot\vec{\omega}_c{}^{(2)}/q_e c^2]^{1/2}, \quad (137\text{a})$$

$$\gamma'|_{q^4}\approx\{(1+q^2)+2\lambda q^2[\omega_c/\omega(\gamma)]+(q^2\omega_c/2\omega)\}^{1/2}. \quad (137\text{b})$$

The final radial equation for the axial $\vec{\text{B}}_z$ field generation is, in the paraxial and stationary approximations for the pulse electric field

$$d_r(r\omega_c')/r\approx\{-\lambda(\omega_{\text{pe}}/\omega)^2[d_r(r\eta'^2 q^2/\gamma^2]/2r\}+\{-\omega_c' d_r(\gamma')/\gamma)/(1+\omega_c^2/\omega_{\text{pe}}^2/\gamma)\} \quad (138)$$

$$\omega_c'=\omega_c/\omega.$$

In the rhs of Eq. (138) the first term enclosed in curly brackets is the IFE effect without n_0 density gradient, while the second term enclosed in curly brackets is the *dressed* ponderomotive source term coming from the current $\vec{\text{J}}_{p_2}$. We have neglected here the contribution of $\vec{\text{B}}_p=\text{curl}(\vec{\text{A}}_p)$ that enters in the current $\vec{\text{J}}_{p_1}$.

The complete set of equations is composed of (i) Eq. (138); (ii) Eq. (128) for (Σ); (iii) Eq. (135b) for (Φ); (iv) relation (125) between n and Φ (or Ψ'); and (v) the chosen closure (133) $\vec{\Omega}\approx\vec{\Omega}'\approx m_0\omega_c$.

In the Appendix, simple analytical formulas for $\vec{\text{B}}_S$ are given in the nonfeedback case when γ and Σ are considered as fixed functions of r only.

C. Simple estimates for $\langle\vec{\text{B}}\rangle$ strength from Eq. (138)

The $\langle\vec{\text{B}}\rangle$ amplitudes are strongly pulse model-dependent (shape of the pulse, duration, intensity, focalizing parameter $k_p r_0, r_0$ being the electric-field transverse gradient length). We can make the following estimates from Eq. (138):

$$\langle\vec{B}_z\rangle/\vec{B}_c\Leftrightarrow\omega_c/\omega,\quad B_c=(m_0\omega/e)$$

magnetization M contribution dominant

$$\omega_c/\omega\approx(\omega_{\text{pe}}/\omega)^2(\text{I}/4\text{I}_c)/(1+\text{I}/\text{I}_c), \quad (139)$$

density gradient term $C(\vec{\nabla}n_0)$ dominant

$$\omega_c/\omega\approx(\omega_{\text{pe}}/\omega)^2/2(\text{I}/\text{I}_c)/(1+\text{I}/\text{I}_c), \quad (139')$$

ponderomotive source (circular case) dominant

$$\omega_c/\omega=2^{1/2}[\omega_{\text{pe}}(0)/\omega_0]\{1+[I(r=0)/I_c]\}^{1/2}-\{1+[I(r)/I_c]^{1/2}\}^{1/2}, \quad (140\text{a})$$

$$\rightarrow|\omega_c/\omega|<2^{1/2}(\omega_{\text{pe}}/\omega_0)\{1+[I(0)/I_c]\}^{1/4}, \quad (140\text{b})$$

$$\rightarrow|\omega_c/\omega|<\{(\omega_{\text{pe}}/\omega_0)[I(0)/I_c]\}^{1/2}. \quad (140\text{c})$$

Where we have used in Eq. (140a) the approximation for γ: $\gamma=(1+q^2)^{1/2}$; Eq. (140b) is taken in the ultrarelativistic limit of (140a) and (140c) is an upper bound for $|\langle\vec{\text{B}}\rangle|$ in the classical limit (there are conditions connecting the parameters $k_p r_0$ and I/I_c, see, for example, Ref. [20] and therein). One finds with Eq. (139) the usual $|\langle\vec{\text{B}}\rangle|$ dependence in q^2 while Eq. (140c) gives a law in q and Eq. (140b) a $q^{1/2}$ law. Numerical application for $k_p r_0=10$, $\text{I}/\text{I}_c=1$, $n_0=10^{19}\text{ cm}^{-3}$, $\lambda=1\ \mu\text{m}$, $\omega_{\text{pe}}/\omega_0=1/4$, we find $|\langle\vec{\text{B}}\rangle|=60$ MG.

D. The numerical procedure

We solve the system (i) to (v) above, with the following boundary conditions: $\lim_{r\to\infty}\omega_c=0$, $\lim_{r\to\infty}\Phi=0$, $\lim_{r\to\infty}\Sigma=0$, and $\lim_{r\to\infty}\partial_r\Sigma=0$. We choose a second-order Runge-Kutta finite-difference scheme integrating inward from an outer radius $r_{\max}$ for which the previous conditions hold. An asymptotic connection at $r=r_{\max}$ is provided in order to match the vacuum solution for the electric field (or vector potential) with the running Σ profile for $r<r_{\max}$.

The vacuum solution could be chosen assuming a radial Gaussian profile for Σ as an initial condition. Another condition consists of regularizing the solution at $r=0$ by imposing $d\Sigma(0)/dr=0$ and the vanishing of Σ at infinity [21]. The parameters q (energy strength) and $k_p r_o$ (initial beam radius normalized) are fixed at $r=r_{\max}$ with the initial Gaussian condition $\Sigma(r_{\max})=qe^{-(\rho/k_p r_0)^2}$, with $\rho=k_p r$ as a normalized variable.

In another simulation we use as an initial condition is the asymptotic formula for Σ: $\Sigma(\rho_{\max})=qe^{-(\kappa\rho)}/(\kappa\rho)^{1/2}$. This last expression comes from the modified Bessel function $(q)K_0(\kappa\rho)$ verifying the differential equation [from Eq. (128)] $\Delta_{\rho\rho}\Sigma-(1-\sigma)\Sigma=0$, with $0<\sigma<1$, $\kappa^2=1-\sigma$.

When there is electron cavitation $\langle n\rangle=0$ and following the prescription of Ref. [21], we solve then the equation $\Delta_{\rho\rho}\Sigma+\sigma\Sigma=0$, whose solution is $(q)J_0(\sigma^{1/2}\rho)$. The choice of the boundary conditions is, however, very important when dealing with nonlinear equations.

E. The results and physical discussion

There are three regimes reminiscent of the RSF unmagnetized process, P^* being the threshold power for RSF [22], depending on the ratio $P/P^*<1$, $=1$ or >1 as subcritical, critical, and overcritical regimes, with a given parameter δ ranging from low-density $\delta\ll 1$ to dense plasmas $\delta=1$.

To compute P^* more precisely we have the known result [22] for $\vec{B}=\vec{0}$, $P^*=P_0/\delta^2$ and from the dispersion relation (see, also, [21])

$$N^2=(kc/\omega)^2\approx 1+\omega_p^2/\omega^2\gamma(q),$$

$$\sigma\approx 1/\gamma(q),\quad \gamma(q)\approx(1+q^2/2)^{1/2}. \tag{141}$$

In this last equation, we see that $\sigma^*(P^*)$ is a decreasing function of q. Whereas for the finite $\langle\vec{B}\rangle$ field using the dispersion relation in the magnetized case, we have an estimate of σ as

$$N^2=(kc/\omega)^2\approx 1+(\omega_p^2/\omega^2\gamma)[1/(1-\lambda\,\omega_c/\omega\gamma)],$$

$$\sigma\approx\eta/\gamma,\gamma(q,\omega_c). \tag{142}$$

Thus, if ω_c is strong, η is high, and the threshold power both for self focusing (and for cavitation) should be reduced *a priori*. For a review on the RSF process in the unmagnetized case, see Refs. [23].

Figures 1 and 2 show, respectively, the electric-field and density radial profiles in the three cases $\sigma/\sigma^*>1,=1$ or <1, without B_s and using the asymptotic value for the solution in K_0.

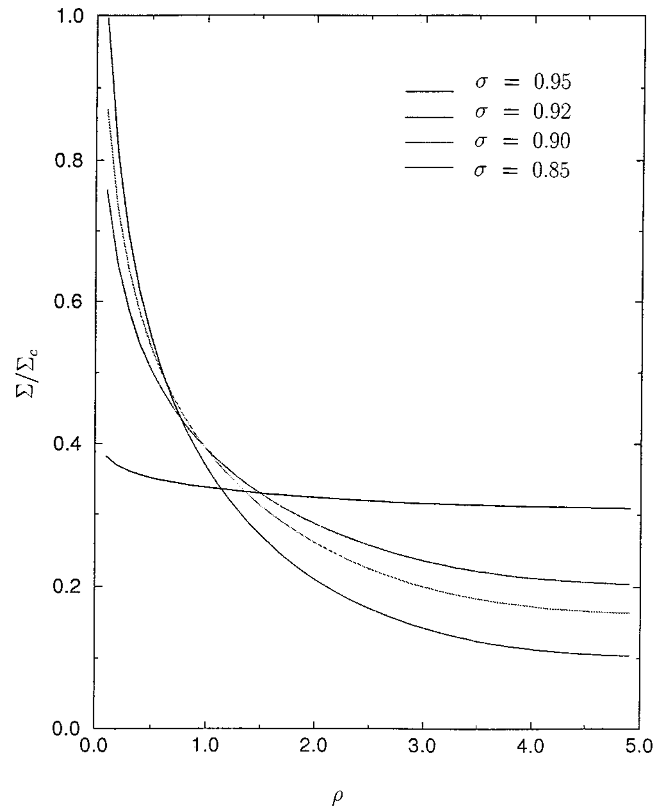

FIG. 1. Electric-field Σ (normalized to the Compton electric-field Σ_c) radial profile for $\vec{B}_S=\vec{0}$, parameters are $q=1$, $k_p r_0=1$, $\delta=0.1$ and for various values of σ subcritical, around critical and supercritical, see text for the choice of initial conditions.

Note that the electric-field profile is modified (increased by cavitation near $r=0$ and also outwards by the edge condition when increasing the value of κ). The electric-field profile is only slightly modified here since the B_s field is weak, and so is the density profile, as compared to the unmagnetized case. However, in Figs. 3–5, we show the density profile in the case of various magnetic fields, first taking into account the total magnetic sources (terms I and II) in Eq. (138). We define term I as the nonself-consistent source (NSC) obtained considering the inverse Faraday effect only, while term II corresponds to the self-consistent (SC) magnetic source computed in this paper. We see apparently no difference between the three situations with dominant NSC, SC, or (NSC+SC) sources *a priori*. In Figs. 6–8 are plotted the magnetic fields profiles in these three cases for the three selected values of σ above, around, and below the critical value σ^*. Here, the difference between the different induced-magnetic fields becomes more obvious. Note that the total (NSC+SC) source gives a B_s field that is not merely the sum of the two cases (SC, NSC) because of the nonlinear dependence of the sources terms in ω_c.

To enhance the different magnetic sources contributions more clearly, (see also the Appendix), we show in Fig. 9 the magnetic-field profile using a fixed Gaussian Σ profile solv-

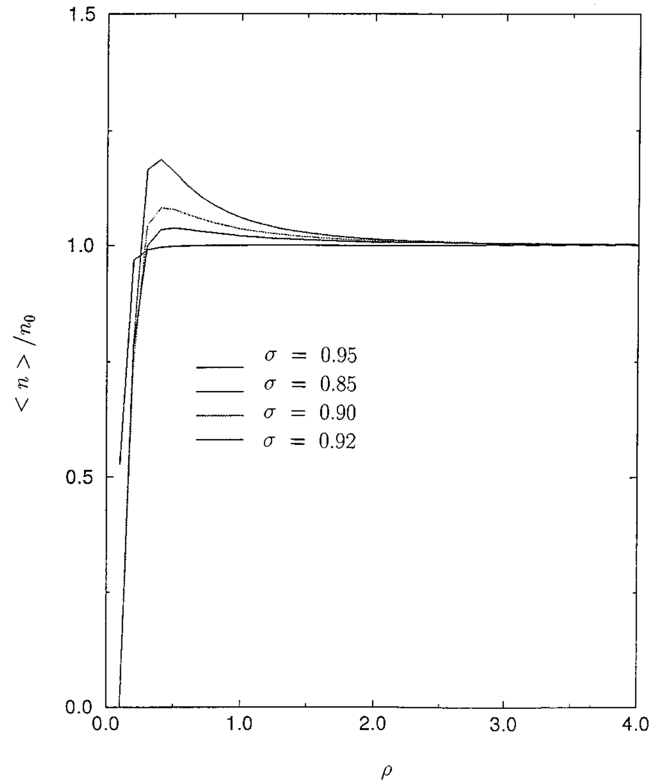

FIG. 2. Density $\langle n\rangle/n_0$ radial profile for the same parameters and σ values than in Fig. 1.

ing Eq. (138) for B_s and for $\langle n\rangle$ through the potential equation. In Fig. 10, we select a case with stronger B_s (δ is increased) and in Fig. 11, we show the corresponding density profile with the tendency of B_s to prevent the cavitation of $\langle n\rangle$.

Our main numerical conclusions are (i) a lowering of the threshold power with ω_c for RSF and (ii) a decrease of the cavitation for n and a limitation of the maximum intensity in the dug channel that remains partially matter filled, hence, confinement of matter is expected and the light beam remains well self focused in the presence of magnetic effects.

V. MAIN CONCLUSIONS

We have given an extended theory of magnetic-field self generation in relativistic cold fluid plasmas. Performing this study relies first on the evaluation of the relevant current sources. We have derived vectorial equations coupling the pump electromagnetic pulse with its self-induced collective fluctuations of density and of magnetic field. We have selected an important application leading to magnetic guiding of light in the case of underdense plasmas, extending the description of the relativistic self-focusing process to the self-magnetized case. The undertaken numerical simulations have allowed us to see the effects of every nonlinear source contribution to the generation of quasistatic magnetic field.

The influence of the self-generated B field on the propagation of the pump is believed to be important and various indications in recent experiments and numerical studies are confirming this point. Especially in overdense plasmas with

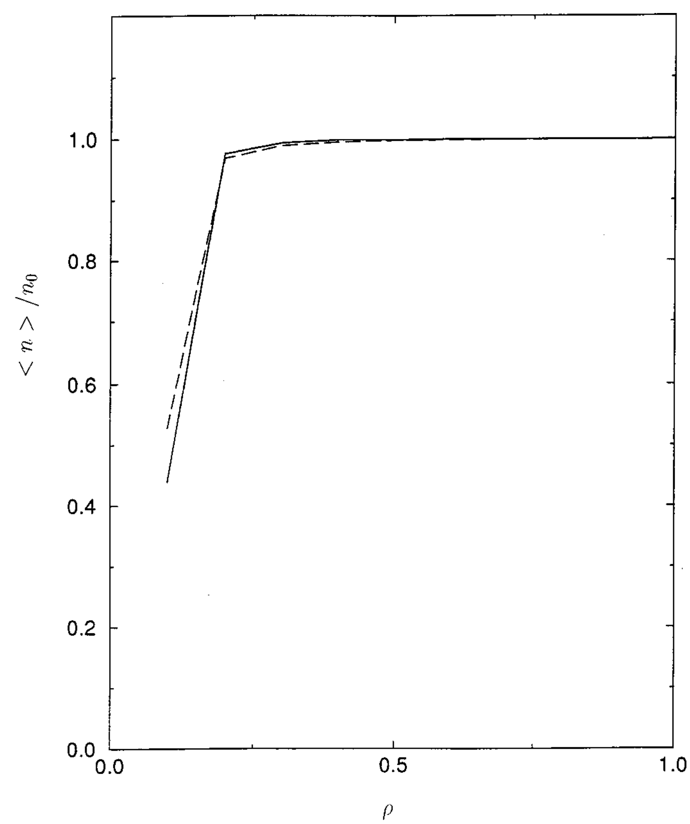

FIG. 3. Same conditions as in Fig. 2 for density n, for $\sigma = 0.95$ (above cavitation threshold) but with different magnetic sources: — — — (total ω_c), · · · · (ω_c from NSC term only),— — (ω_c from SC term only).

wave propagation by the mechanism (still unclear) of "self-induced transparency" the interaction between the pump wave and the quasistatic $\vec{B}$ field is believed to play a crucial role.

Our formalism has to be extended to investigate more specifically the non-QSA limit and also the pump nonlinear propagation in the overdense plasma situation by keeping the suitable fast-source currents at the pump harmonic frequencies. These studies deserve future work.

APPENDIX: SELF-MAGNETIC FIELD GENERATION: SIMPLE ANALYTICAL SOLUTIONS IN VARIOUS LIMITING CASES

The main approximation here is to consider that the pump electric field remains fixed (no feedback of perturbations, not to be confused with the nonself-consistent case, see main text).

1. Magnetization M dominant (inverse Faraday effect)

We start from the generation equation for quasistatic magnetic field to find

Since $\mathrm{curl}(\vec{\omega}_c)\approx\mathrm{curl}(\vec{\mathrm{M}}),\vec{\nabla}\cdot(\vec{\omega}_c)=0:\vec{\omega}_c\approx\vec{\mathrm{M}}$,

$$(\vec{\omega}_c/\omega)=(\mu_0 q_e/m_0\omega)\vec{\mathrm{M}},\quad \text{for}\quad \vec{\nabla}(n_0)=0,$$

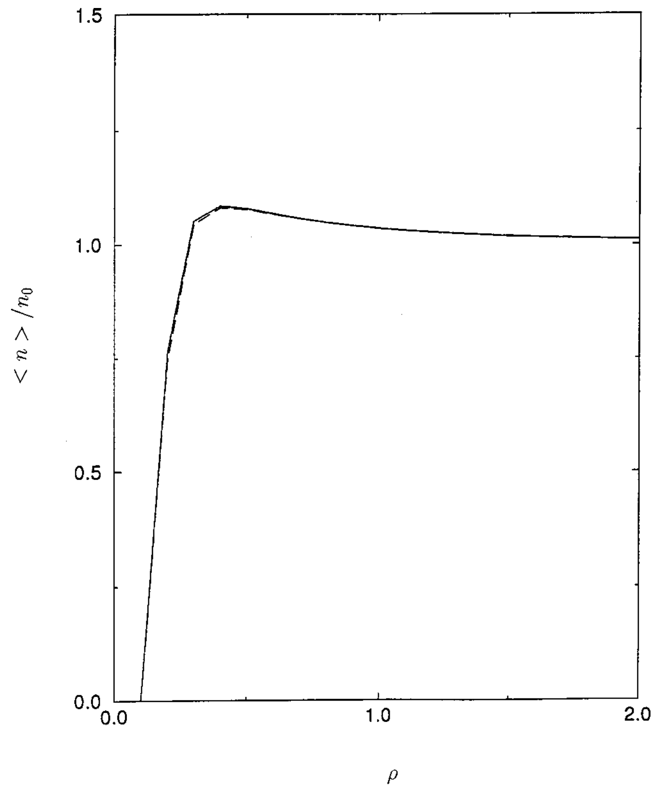

FIG. 4. Same conditions as in Fig. 3 for n, for $\sigma = 0.90$ (around cavitation threshold) always with the different magnetic fields.

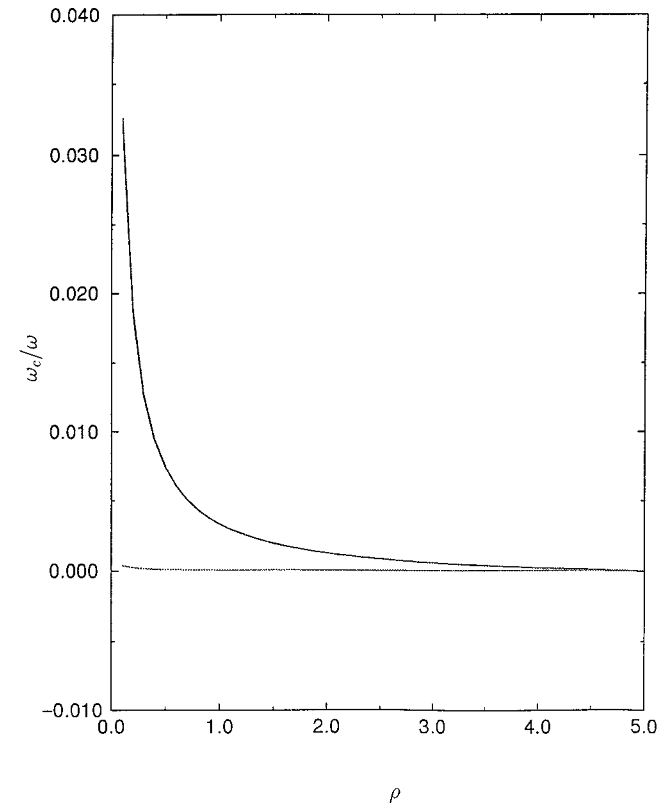

FIG. 6. Magnetic field (normalized to frequency ω) radial profile with the three possible sources as in Fig. 3, for $\sigma = 0.9$.

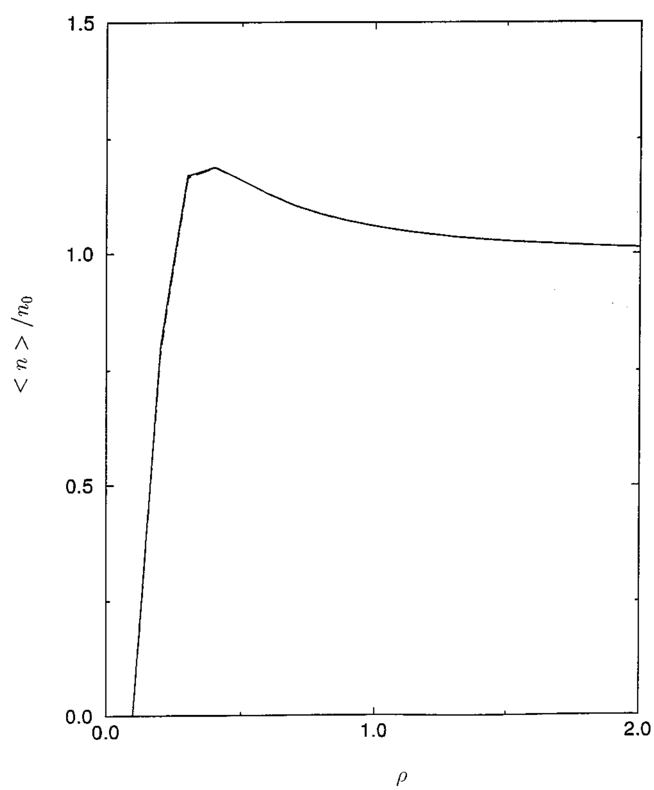

FIG. 5. Same conditions than in Fig. 3 for n, for $\sigma = 0.85$ (under cavitation threshold) always with the different magnetic fields.

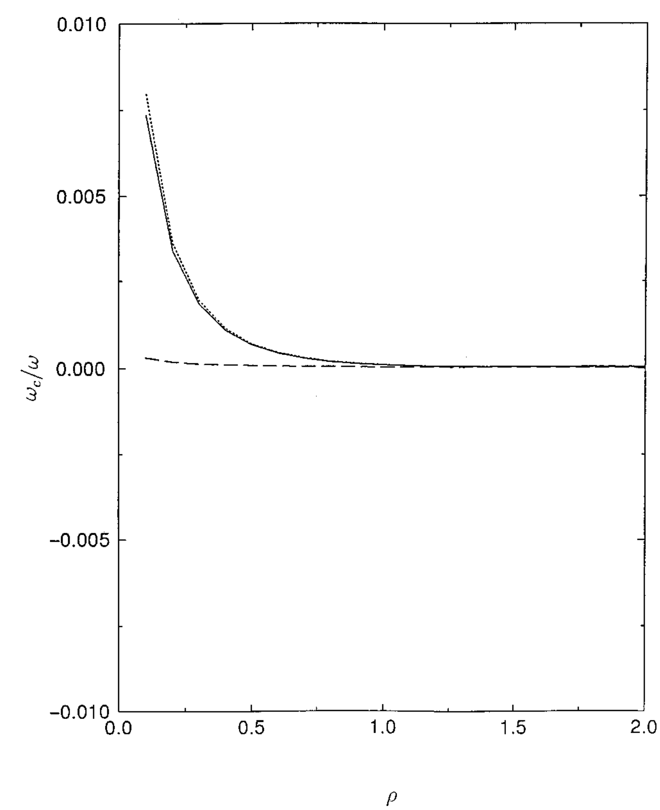

FIG. 7. Magnetic-field radial profile as in Fig. 6 but with $\sigma = 0.9$.

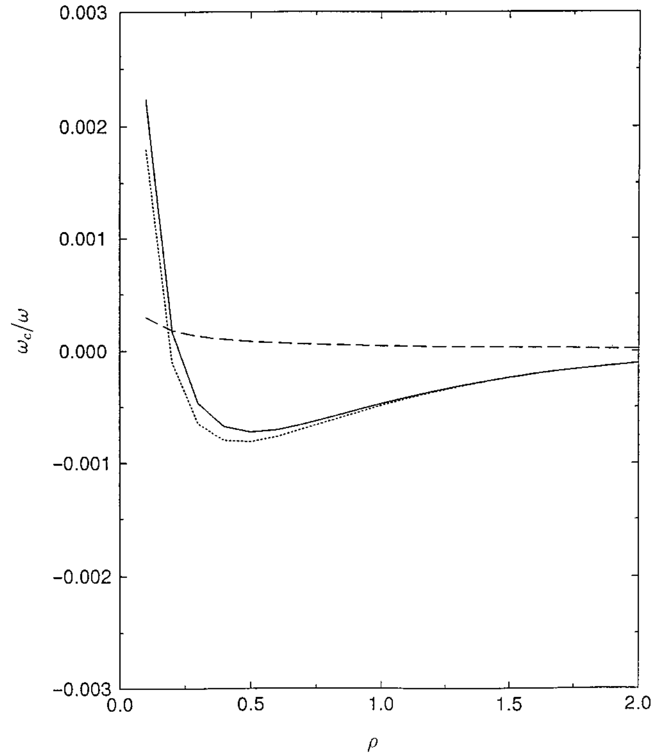

FIG. 8. Magnetic-field radial profile as in Fig. 7 but with $\sigma = 0.85$.

$$(\vec{\omega}_c/\omega)=2(\mu_0 q_e/m_0\omega)\vec{\mathrm{M}},\quad \text{for } \vec{\nabla}(n_0)\neq\vec{0},$$

$$(\omega_c/\omega)=(\omega_{\mathrm{pe}}^2/2\omega^2)(\eta'^2 q^2/\gamma^2),$$

$$(\omega_c^{(2)}/\omega)\approx q^2(\omega_{\mathrm{pe}}^2/2\omega^2)(/\gamma^2).$$

2. Ponderomotive term (in $\vec{\mathbf{F}}_p\times\langle\vec{\mathbf{V}}\rangle$) dominant

With the choice of closure here $\Omega'/m_0=\omega_c+(2\omega_{\mathrm{cp}})$,

$$(1/r)\partial_r(r\omega_c)\approx-\omega_{\mathrm{pe}}^2(\langle n\rangle/n_0)\{(\omega_c+2\omega_{\mathrm{cp}})[d_r(\gamma')]/\gamma)\times[\omega_{\mathrm{pe},0}^2+(\omega_c+2\omega_{\mathrm{cp}})^2/\gamma]\},$$

to order q^2, for $\omega_c\gg2\omega_{\mathrm{cp}}$.

For the general case (GC)

$$d_r(r\omega_c/\omega)/r\approx-\{(\omega_{\mathrm{pe}}/\omega)^2\{(2)d_r[n_0 r(\gamma'/\gamma)^2|\Sigma|^2]/n_0 r\}-\{(\omega_c d_r(\gamma')/\omega\gamma)/(1+\omega_c^2/\omega_{\mathrm{pe}}^2\gamma)\}$$

For the sub case where the second term enclosed in curly brackets is dominant (GC 2)

$$d_r(r\omega_c/\omega)/r\approx-\{[\omega_c d_r(\gamma')/\omega\gamma]/(1+\omega_c^2/\omega_{\mathrm{pe}}^2\gamma)\},$$

and special subcases

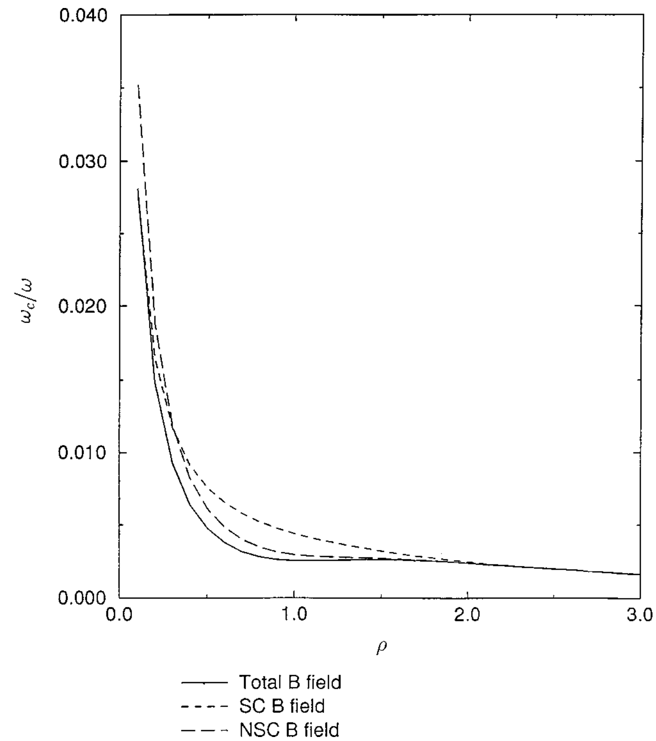

FIG. 9. Magnetic-field radial profile, with same parameters as in Fig. 1, but with a fixed Gaussian electric-field radial profile.

(1) for $\omega_c^2\ll\omega_{\mathrm{pe}}^2\gamma$,

$$(1/r)\partial_r(r\omega_c)\approx-\omega_c[d_r(\gamma')]/\gamma\approx-\omega_c[d_r(\gamma)]/\gamma,$$

$(\omega_c/\omega_{\mathrm{cl}})=[\gamma_1 r_1/\gamma(r)r]$ with proper choice of constants γ_1,r_1,

(2) for $\omega_c^2\gg\omega_{\mathrm{pe}}^2\gamma$,

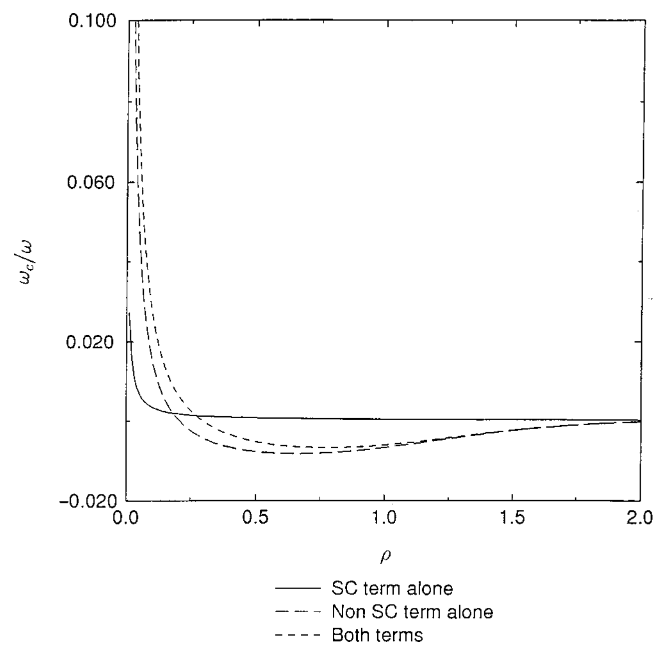

FIG. 10. Magnetic-field radial profile, same conditions as in Fig. 9, but with $\delta=0.4$.

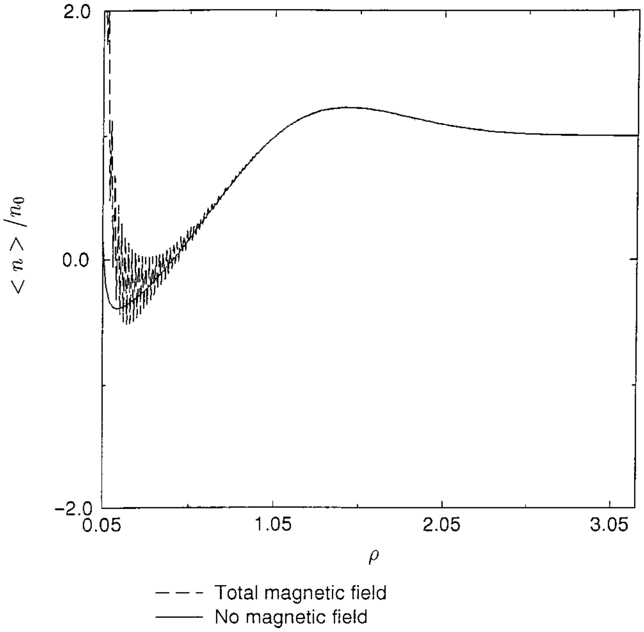

FIG. 11. Effect of total B_s on the density profile for conditions in Fig. 10.

$$(1/r)\partial_r(r\omega_c) \approx -\omega_{\rm pe}^2 d_r(\gamma')/\omega_c,$$

$$\omega_c = (\omega_{\rm pe}/r)\left(2\int_0^r -r^2 d_r(\gamma')dr\right)^{1/2},$$

with $\omega_c(r=0)=0$.

For GC 2, there is no simple solution (see numerical solution). For an expansion in q^2, by taking $\gamma=$const in the denominator:

$$d_r(r\omega_c)(1+\omega_c^2/\omega_{\rm pe}^2\gamma_0)/(\omega_c r) \approx -[d_r(\gamma')/\gamma] \approx -d\ln(\gamma),$$

$$\omega_c = (\omega_{\rm pe}/r)\left(2\int_0^r -r^2 d_r(\ln\gamma^{(')}/r\omega_c)dr\right)^{1/2},$$

$$\text{or}\ \ [r\omega_c\gamma_1/r_1\gamma(r)\omega_c] = \exp\left(\int_0^r r^{-2}d_r(r\omega_c)^2 dr/2\omega_{\rm pe}^2\right).$$

Comparison of terms.

$$(1)\ \ \text{for}\,\omega_c^2 \ll \omega_{\rm pe}^2\gamma(\omega_c/\omega_{c1}/[(\omega_c(M)/\omega_c] = 1/(\gamma^{(2)}r)$$
$$\times(\delta^2 q^2/2\gamma^2);\quad (2)\ \ \text{for}\,\omega_c^2 \gg \omega_{\rm pe}^2\gamma,$$

$$\omega_c^2/\omega_c^2(M) = 2[(\omega_{\rm pe}^2/r^2(\delta^2q^2/2\gamma')^2]\left(\int_0^r r^2 d_r(\gamma')dr\right).$$

Remark. The solutions are found above in the nonfeedback case only, i.e., for Σ and γ are given as fixed functions of r, and are assumed to be independent of ω_c. When the magnetic coupling is set on, we have $\gamma(\Sigma(r,\omega_c)\omega_c,r)$ and the solutions for B_s could be obtained only numerically, which is the purpose of Sec. IV.

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